Chapter 13

Differential Models

This chapter provides a continuous-time counterpart to the state transition equation, $x_{k+1} = f(x_k, u_k)$, which was crucial in Chapter 2. On a continuous state space, X (assumed to be a smooth manifold), it will be defined as $\dot{x} = f(x, u)$, which intentionally looks similar to the discrete version. It will still be referred to as a state transition equation. It will also be called a *system* (short for *control system*), which is a term used in control theory. There are no obstacle regions in this chapter. Obstacles will appear again when planning algorithms are covered in Chapter 14. In continuous time, the state transition function $f(x, u)$ yields a velocity as opposed to the next state. Since the transitions are no longer discrete, it does not make sense to talk about a "next" state. Future states that satisfy the differential constraints are obtained by integration of the velocity. Therefore, it is natural to specify only velocities. This relies on the notions of tangent spaces and vector fields, as covered in Section 8.3.

This chapter presents many example models that can be used in the planning algorithms of Chapter 14. Section 13.1 develops differential constraints for the case in which X is the C-space of one or more bodies. These constraints commonly occur for wheeled vehicles (e.g., a car cannot move sideways). To represent dynamics, constraints on acceleration are needed. Section 13.2 therefore introduces the *phase space*, which enables any problem with dynamics to be expressed as velocity constraints on an enlarged state space. This collapses the higher order derivatives down to being only first-order, but it comes at the cost of increasing the dimension of the state space. Section 13.3 introduces the basics of Newton-Euler mechanics and concludes with expressing the dynamics of a free-floating rigid body. Section 13.4 introduces some concepts from advanced mechanics, including the Lagrangian and Hamiltonian. It also provides a model of the dynamics of a kinematic chain of bodies, which applies to typical robot manipulators. Section 13.5 introduces differential models that have more than one decision maker.

13.1 Velocity Constraints on the Configuration Space

In this section, it will be assumed that $X = \mathcal{C}$, which is a C-space of one or more rigid bodies, as defined in Section 4.2. Differential models in this section are all expressed as constraints on the set of allowable velocities at each point in $\mathcal{C}$. This results in first-order differential equations because only velocities are constrained, as opposed to accelerations or higher order derivatives.

To carefully discuss velocities, it will be assumed that $\mathcal{C}$ is a smooth manifold, as defined in Section 8.3.2, in addition to a topological manifold, as defined in Section 4.1.2. It may be helpful to keep the cases $\mathcal{C} = \mathbb{R}^2$ and $\mathcal{C} = \mathbb{R}^3$ in mind. The velocities are straightforward to define without resorting to manifold technicalities, and the dimension is low enough that the concepts can be visualized.

13.1.1 Implicit vs. Parametric Representations

There are two general ways to represent differential constraints: parametric and implicit. Many parallels can be drawn to the parametric and implicit ways of specifying functions in general. Parametric representations are generally easier to understand and are simpler to use in applications. Implicit representations are more general but are often more difficult to utilize. The intuitive difference is that implicit representations express velocities that are *prohibited*, whereas parametric representations directly express the velocities that are *allowed*. In this chapter, a parametric representation is obtained wherever possible; nevertheless, it is important to understand both.

13.1.1.1 Implicit representation

The planar case For purposes of illustration, suppose that $\mathcal{C} = \mathbb{R}^2$. A configuration is expressed as $q = (x, y) \in \mathbb{R}^2$, and a velocity is expressed as $(\dot{x}, \dot{y})$. Each $(\dot{x}, \dot{y})$ is an element of the tangent space $T_q(\mathbb{R}^2)$, which is a two-dimensional vector space at every (x, y). Think about the kinds of constraints that could be imposed. At each $q \in \mathbb{R}^2$, restricting the set of velocities yields some set $U(q) \subset T_q(\mathbb{R}^2)$. The parametric and implicit representations will be alternative ways to express $U(q)$ for all $q \in \mathbb{R}^2$.

Here are some interesting, simple constraints. Each yields a definition of $U(q)$ as the subset of $T_q(\mathbb{R}^2)$ that satisfies the constraints.

1. $\dot{\boldsymbol{x}} > \mathbf{0}$**:** In this case, imagine that you are paddling a boat on a swift river that flows in the positive x direction. You can obtain any velocity you like in the y direction, but you can never flow against the current. This means that all integral curves increase monotonically in x over time.

2. $\dot{\boldsymbol{x}} \geq \mathbf{0}$**:** This constraint allows you to stop moving in the x direction. A velocity perpendicular to the current can be obtained (for example, $(0, 1)$

causes motion with unit speed in the positive y direction).

3. $\dot{x} > 0$, $\dot{y} > 0$: Under this constraint, integral curves must monotonically increase in both x and y.

4. $\dot{x} = 0$: In the previous three examples, the set of allowable velocities remained two-dimensional. Under the constraint $\dot{x} = 0$, the set of allowable velocities is only one-dimensional. All vectors of the form $(0, \dot{y})$ for any $\dot{y} \in \mathbb{R}$ are allowed. This means that no motion in the x direction is allowed. Starting at any (x_0, y_0), the integral curves will be of the form $(x_0, y(t))$ for all $t \in [0, \infty)$, which confines each one to a vertical line.

5. $a\dot{x} + b\dot{y} = 0$: This constraint is qualitatively the same as the previous one. The difference is that now the motions can be restricted along any collection of parallel lines by choosing a and b. For example, if $a = b = 1$, then only diagonal motions are allowed.

6. $a\dot{x} + b\dot{y} + c = 0$: This constraint is similar to the previous one, however the behavior is quite different because the integral curves do not coincide. An entire half plane is reached. It also impossible to stop becasue $\dot{x} = \dot{y} = 0$ violates the constraint.

7. $\dot{x}^2 + \dot{y}^2 \leq 1$: This constraint was used in Chapter 8. It has no effect on the existence of solutions to the feasible motion planning problem because motion in any direction is still allowed. The constraint only enforces a maximum speed.

8. $\dot{x}^2 + \dot{y}^2 \geq 1$: This constraint allows motions in any direction and at any speed greater than 1. It is impossible to stop or slow down below unit speed.

Many other constraints can be imagined, including some that define very complicated regions in $\mathbb{R}^2$ for each $U(q)$. Ignoring the fact that $\dot{x}$ and $\dot{y}$ represent derivatives, the geometric modeling concepts from Section 3.1 can even be used to define complicated constraints at each q. In fact, the constraints expressed above in terms of $\dot{x}$ and $\dot{y}$ are simple examples of the semi-algebraic model, which was introduced in Section 3.1.2. Just replace x and y from that section by $\dot{x}$ and $\dot{y}$ here.

If at every q there exists some open set O such that $(0, 0) \in O$ and $O \subseteq U(q)$, then there is no effect on the existence of solutions to the feasible motion planning problem. Velocities in all directions are still allowed. This holds true for velocity constraints on any smooth manifold [924].

So far, the velocities have been constrained in the same way at every $q \in \mathbb{R}^2$, which means that $U(q)$ is the same for all $q \in \mathbb{R}^2$. Constraints of this kind are of the form $g(\dot{x}, \dot{y}) \bowtie 0$, in which $\bowtie$ could be $=$, $<$, $>$, $\leq$, or $\geq$, and g_i is a function from $\mathbb{R}^2$ to $\mathbb{R}$. Typically, the $=$ relation drops the dimension of $U(x)$ by one, and the others usually leave it unchanged.

Now consider the constraint $\dot{x} = x$. This results in a different one-dimensional set, $U(q)$, of allowable velocities at each $q \in \mathbb{R}^2$. At each $q = (x, y)$, the set of allowable velocities must be of the form $(x, \dot{y})$ for any $\dot{y} \in \mathbb{R}$. This means that as x increases, the velocity in the x direction must increase proportionally. Starting at any positive x value, there is no way to travel to the y-axis. However, starting on the y-axis, the integral curves will always remain on it! Constraints of this kind can generally be expressed as $g(x, y, \dot{x}, \dot{y}) \bowtie 0$, which allows the dependency on x or y.

General configuration spaces Velocity constraints can be considered in the same way on a general C-space. Assume that $\mathcal{C}$ is a smooth manifold (a manifold was not required to be smooth in Chapter 4 because derivatives were not needed there). All constraints are expressed using a coordinate neighborhood, as defined in Section 8.3.2. For expressing differential models, this actually makes an n-dimensional manifold look very much like $\mathbb{R}^n$. It is implicitly understood that a change of coordinates may occasionally be needed; however, this does not complicate the expression of constraints. This makes it possible to ignore many of the manifold technicalities and think about the constraints as if they are applied to $\mathbb{R}^n$.

Now consider placing velocity constraints on $\mathcal{C}$. Imagine how complicated velocity constraints could become if any semi-algebraic model is allowed. Velocity constraints on $\mathcal{C}$ could be as complicated as any $\mathcal{C}_{obs}$. It is not even necessary to use algebraic primitives. In general, the constraints can be expressed as

$$g(q, \dot{q}) \bowtie 0, \tag{13.1}$$

in which $\bowtie$ could once again be $=$, $<$, $>$, $\leq$, or $\geq$. The same expressive power can be maintained even after eliminating some of these relations. For example, any constraint of the form (13.1) can be expressed as a combination of constraints of the form $g(q, \dot{q}) = 0$ and $g(q, \dot{q}) < 0$. All of the relations are allowed here, however, to make the formulations simpler.

Constraints expressed in the form shown in (13.1) are called *implicit*. As explained in Chapters 3 and 4, it can be very complicated to obtain a parametric representation of the solutions of implicit equations. This was seen, for example, in Section 4.4, in which it was difficult to characterize the set of configurations that satisfy closure constraints. Nevertheless, we will be in a much better position in terms of developing planning algorithms if a parametric representation of the constraints can be obtained. Fortunately, most constraints that are derived from robots, vehicles, and other mechanical systems can be expressed in parametric form.

13.1.1.2 Parametric constraints

The parametric way of expressing velocity constraints gives a different interpretation to $U(q)$. Rather than directly corresponding to a velocity, each $u \in U(q)$

is interpreted as an abstract action vector. The set of allowable velocities is then obtained through a function that maps an action vector into $T_q(\mathcal{C})$. This yields the *configuration transition equation* (or *system*)

$$\dot{q} = f(q, u), \tag{13.2}$$

in which f is a continuous-time version of the state transition function that was developed in Section 2.1. Note that (13.2) actually represents n scalar equations, in which n is the dimension of $\mathcal{C}$. The system will nevertheless be referred to as a single equation in the vector sense. Usually, $U(q)$ is fixed for all $q \in \mathcal{C}$. This will be assumed unless otherwise stated. In this case, the fixed action set is denoted as U.

There are two interesting ways to interpret (13.2):

1. **Subspace of the tangent space:** If q is fixed, then f maps from U into $T_q(\mathcal{C})$. This parameterizes the set of allowable velocities at q because a velocity vector, $f(q, u)$, is obtained for every $u \in U(q)$.

2. **Vector field:** If u is fixed, then f can be considered as a function that maps each $q \in \mathcal{C}$ into $T_q(\mathcal{C})$. This means that f defines a vector field over $\mathcal{C}$ for every fixed $u \in U$.

Example 13.1 (Two Interpetations of $\dot{q} = f(q, u)$) Suppose that $\mathcal{C} = \mathbb{R}^2$, which yields a two-dimensional velocity vector space at every $q = (x, y) \in \mathbb{R}^2$. Let $U = \mathbb{R}$, and $\dot{q} = f(q, u)$ be defined as $\dot{x} = u$ and $\dot{y} = x$.

To obtain the first interpretation of $\dot{q} = f(q, u)$, hold $q = (x, y)$ fixed; for each $u \in U$, a velocity vector $(\dot{x}, \dot{y}) = (u, x)$ is obtained. The set of all allowable velocity vectors at $q = (x, y)$ is

$$\{(\dot{x}, \dot{y}) \in \mathbb{R}^2 \mid \dot{y} = x\}. \tag{13.3}$$

Suppose that $q = (1, 2)$. In this case, any vector of the form $(u, 1)$ for any $u \in \mathbb{R}$ is allowable.

To obtain the second interpretation, hold u fixed. For example, let $u = 1$. The vector field $(\dot{x}, \dot{y}) = (1, x)$ over $\mathbb{R}^2$ is obtained. ■

It is important to specify U when defining the configuration transition equation. We previously allowed, but discouraged, the action set to depend on q. Any differential constraints expressed as $\dot{q} = f(q, u)$ for any U can be alternatively expressed as $\dot{q} = u$ by defining

$$U(q) = \{\dot{q} \in \mathbb{R}^n \mid \exists u \in U \text{ such that } \dot{q} = f(q, u)\} \tag{13.4}$$

for each $q \in \mathcal{C}$. In this definition, $U(q)$ is not necessarily a subset of U. It is usually more convenient, however, to use the form $\dot{q} = f(q, u)$ and keep the same U for all q. The common interpretation of U is that it is a set of fixed actions that can be applied from any point in the C-space.

In the context of ordinary motion planning, a configuration transition equation did not need to be specifically mentioned. This issue was discussed in Section 8.4. Provided that U contains an open subset that contains the origin, motion in any direction is allowed. The configuration transition equation for basic motion planning was simply $\dot{q} = u$. Since this does not impose constraints on the direction, it was not explicitly mentioned. For the coming models in this chapter, constraints will be imposed on the velocities that restrict the possible directions. This requires planning algorithms that handle differential models and is the subject of Chapter 14.

13.1.1.3 Conversion from implicit to parametric form

There are trade-offs between the implicit and parametric ways to express differential constraints. The implicit representation is more general; however, the parametric form is more useful because it explicitly gives the possible actions. For this reason, it is often desirable to derive a parametric representation from an implicit one. Under very general conditions, it is theoretically possible. As will be explained shortly, this is a result of the implicit function theorem. Unfortunately, the theoretical existence of such a conversion does not help in actually performing the transformations. In many cases, it may not be practical to determine a parametric representation.

To model a mechanical system, it is simplest to express constraints in the implicit form and then derive the parametric representation $\dot{q} = f(q, u)$. So far there has been no appearance of u in the implicit representation. Since u is interpreted as an action, it needs to be specified while deriving the parametric representation. To understand the issues, it is helpful to first assume that all constraints in implicit form are linear equations in $\dot{q}$ of the form

$$g_1(q)\dot{q}_1 + g_2(q)\dot{q}_2 + \cdots + g_n(q)\dot{q}_n = 0, \tag{13.5}$$

which are called *Pfaffian constraints*. These constraints are linear only under the assumption that q is known. It is helpful in the current discussion to imagine that q is fixed at some known value, which means that each of the $g_i(q)$ coefficients in (13.5) is a constant.

Suppose that k Pfaffian constraints are given for $k \leq n$ and that they are linearly independent.[1] Recall the standard techniques for solving linear equations. If $k = n$, then a unique solution exists. If $k < n$, then a continuum of solutions exists, which forms an $(n - k)$-dimensional hyperplane. It is impossible to have $k > n$ because there can be no more than n linearly independent equations.

If $k = n$, only one velocity vector satisfies the constraints for each $q \in \mathcal{C}$. A vector field can therefore be derived from the constraints, and the problem is not interesting from a planning perspective because there is no choice of velocities. If $k < n$, then $n - k$ components of $\dot{q}$ can be chosen independently, and then

[1] If the coefficients are placed into an $k \times n$ matrix, its rank is k.

the remaining k are computed to satisfy the Pfaffian constraints (this can be accomplished using linear algebra techniques such as singular value decomposition [399, 961]). The components of $\dot{q}$ that can be chosen independently can be considered as $n-k$ scalar actions. Together these form an $(n-k)$-dimensional action vector, $u = (u_1, \ldots, u_{n-k})$. Suppose without loss of generality that the first $n-k$ components of $\dot{q}$ are specified by u. The configuration transition equation can then be written as

$$\begin{aligned} \dot{q}_1 &= u_1 & \dot{q}_{n-k+1} &= f_{n-k+1}(q, u) \\ \dot{q}_2 &= u_2 & \dot{q}_{n-k+2} &= f_{n-k+2}(q, u) \\ &\vdots & &\vdots \\ \dot{q}_{n-k} &= u_{n-k} & \dot{q}_n &= f_n(q, u), \end{aligned} \tag{13.6}$$

in which each f_i is a linear function of u and is derived from the Pfaffian constraints after substituting u_i for $\dot{q}_i$ for each i from 1 to $n-k$ and then solving for the remaining components of $\dot{q}$. For some values of q, the constraints may become linearly dependent. This only weakens the constraints, which means the dimension of u can be increased at any q for which independence is lost. Such points are usually isolated and will not be considered further.

Example 13.2 (Pfaffian Constraints) Suppose that $\mathcal{C} = \mathbb{R}^3$, and there is one constraint of the form (13.5)

$$2\dot{q}_1 - \dot{q}_2 - \dot{q}_3 = 0. \tag{13.7}$$

For this problem, $n = 3$ and $k = 1$. There are two action variables because $n - k = 2$. The configuration transition equation is

$$\begin{aligned} \dot{q}_1 &= u_1 \\ \dot{q}_2 &= u_2 \\ \dot{q}_3 &= 2u_1 - u_2, \end{aligned} \tag{13.8}$$

in which the last component was obtained by substituting u_1 and u_2, respectively, for $\dot{q}_1$ and $\dot{q}_2$ in (13.7) and then solving for $\dot{q}_3$.

The constraint given in (13.7) does not even depend on q. The same ideas apply for more general Pfaffian constraints, such as

$$(\cos q_3)\dot{q}_1 - (\sin q_3)\dot{q}_2 - \dot{q}_3 = 0. \tag{13.9}$$

Following the same procedure, the configuration transition equation becomes

$$\begin{aligned} \dot{q}_1 &= u_1 \\ \dot{q}_2 &= u_2 \\ \dot{q}_3 &= (\cos q_3)u_1 - (\sin q_3)u_2. \end{aligned} \tag{13.10}$$

■

The ideas presented so far naturally extend to equality constraints that are not linear in $\dot{x}$. At each q, an $(n-k)$-dimensional set of actions, $U(q)$, is guaranteed to exist if the Jacobian $\partial(g_1, \ldots, g_k)/\partial(\dot{q}_1, \ldots, \dot{q}_n)$ (recall (6.28) or see [508]) of the constraint functions has rank k at q. This follows from the *implicit function theorem* [508].

Suppose that there are inequality constraints of the form $g(q, \dot{q}) \leq 0$, in addition to equality constraints. Using the previous concepts, the actions may once again be assigned directly as $\dot{q}_i = u_i$ for all i such that $1 \leq i \leq n-k$. Without inequality constraints, there are no constraints on u, which means that $U = \mathbb{R}^n$. Since u is interpreted as an input to some physical system, U will often be constrained. In a physical system, for example, the amount of energy consumed may be proportional to u. After performing the $\dot{q}_i = u_i$ substitutions, the inequality constraints indicate limits on u. These limits are expressed in terms of q and the remaining components of $\dot{q}$, which are the variables $\dot{q}_{n-k+1}$, ..., $\dot{q}_n$. For many problems, the inequality constraints are simple enough that constraints directly on U can be derived. For example, if u_1 represents scalar acceleration applied to a car, then it may have a simple bound such as $|u_1| \leq 1$.

One final complication that sometimes occurs is that the action variables may already be specified in the equality constraints: $g(q, \dot{q}, u) = 0$. In this case, imagine once again that q is fixed. If there are k independent constraints, then by the implicit function theorem, $\dot{q}$ can be solved to yield $\dot{q} = f(q, u)$ (although theoretically possible, it may be difficult in practice). If the Jacobian $\partial(f_1, \ldots, f_n)/\partial(u_1, \ldots, u_k)$ has rank k at q, then actions can be applied to yield any velocity on a k-dimensional hyperplane in $T_q(\mathcal{C})$. If $k = n$, then there are enough independent action variables to overcome the constraints. Any velocity in $T_q(\mathcal{C})$ can be achieved through a choice of u. This is true only if there are no inequality constraints on U.

13.1.2 Kinematics for Wheeled Systems

The most common family of examples in robotics arises from wheels that are required to roll in the direction they are pointing. Most wheels are not designed to slide sideways. This imposes velocity constraints on rolling vehicles. As a result, there are usually less action variables than degrees of freedom. Such systems are therefore called *underactuated*. It is interesting that, in many cases, vehicles can execute motions that overcome the constraint. For example, a car can parallel park itself anywhere that it could reach if all four wheels could turn to any orientation. This leads to formal concepts such as *nonholonomic constraints* and *small-time local controllability*, which are covered in Section 15.4.

13.1.2.1 A simple car

One of the easiest examples to understand is the *simple car*, which is shown in Figure 13.1. We all know that a car cannot drive sideways because the back wheels would have to slide instead of roll. This is why parallel parking is challenging. If all four wheels could be turned simultaneously toward the curb, it would be trivial

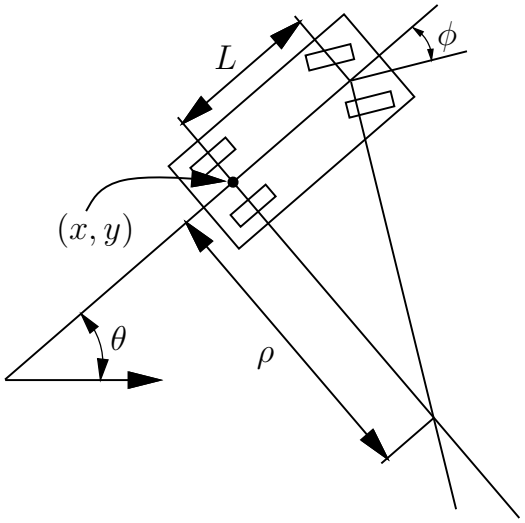


Figure 13.1: The simple car has three degrees of freedom, but the velocity space at any configuration is only two-dimensional.

to park a car. The complicated maneuvers for parking a simple car arise because of rolling constraints.

The car can be imagined as a rigid body that moves in the plane. Therefore, its C-space is $\mathcal{C} = \mathbb{R}^2 \times \mathbb{S}^1$. Figure 13.1 indicates several parameters associated with the car. A configuration is denoted by $q = (x, y, \theta)$. The body frame of the car places the origin at the center of rear axle, and the x-axis points along the main axis of the car. Let s denote the (signed) speed[2] of the car. Let ϕ denote the steering angle (it is negative for the wheel orientations shown in Figure 13.1). The distance between the front and rear axles is represented as L. If the steering angle is fixed at ϕ, the car travels in a circular motion, in which the radius of the circle is ρ. Note that ρ can be determined from the intersection of the two axes shown in Figure 13.1 (the angle between these axes is $|\phi|$).

Using the current notation, the task is to represent the motion of the car as a set of equations of the form

$$\begin{aligned} \dot{x} &= f_1(x, y, \theta, s, \phi) \\ \dot{y} &= f_2(x, y, \theta, s, \phi) \\ \dot{\theta} &= f_3(x, y, \theta, s, \phi). \end{aligned} \tag{13.11}$$

In a small time interval, Δt, the car must move approximately in the direction that the rear wheels are pointing. In the limit as Δt tends to zero, this implies that $dy/dx = \tan\theta$. Since $dy/dx = \dot{y}/\dot{x}$ and $\tan\theta = \sin\theta/\cos\theta$, this condition can

[2]Having a signed speed is somewhat unorthodox. If the car moves in reverse, then s is negative. A more correct name for s would be velocity in the x direction of the body frame, but this is too cumbersome.

be written as a Pfaffian constraint (recall (13.5)):

$$-\dot{x}\sin\theta + \dot{y}\cos\theta = 0. \tag{13.12}$$

The constraint is satisfied if $\dot{x} = \cos\theta$ and $\dot{y} = \sin\theta$. Furthermore, any scalar multiple of this solution is also a solution; the scaling factor corresponds directly to the speed s of the car. Thus, the first two scalar components of the configuration transition equation are $\dot{x} = s\cos\theta$ and $\dot{y} = s\sin\theta$.

The next task is to derive the equation for $\dot{\theta}$. Let w denote the distance traveled by the car (the integral of speed). As shown in Figure 13.1, ρ represents the radius of a circle that is traversed by the center of the rear axle, if the steering angle is fixed. Note that $dw = \rho d\theta$. From trigonometry, $\rho = L/\tan\phi$, which implies

$$d\theta = \frac{\tan\phi}{L} dw. \tag{13.13}$$

Dividing both sides by dt and using the fact that $\dot{w} = s$ yields

$$\dot{\theta} = \frac{s}{L}\tan\phi. \tag{13.14}$$

So far, the motion of the car has been modeled, but no action variables have been specified. Suppose that the speed s and steering angle ϕ are directly specified by the action variables u_s and u_ϕ, respectively. The convention of using a u variable with the old variable name appearing as a subscript will be followed. This makes it easy to identify the actions in a configuration transition equation. A two-dimensional action vector, $u = (u_s, u_\phi)$, is obtained. The configuration transition equation for the simple car is

$$\begin{aligned} \dot{x} &= u_s \cos\theta \\ \dot{y} &= u_s \sin\theta \\ \dot{\theta} &= \frac{u_s}{L}\tan u_\phi. \end{aligned} \tag{13.15}$$

As expressed in (13.15), the transition equation is not yet complete without specifying U, the set of actions of the form $u = (u_s, u_\phi)$. First suppose that any $u_s \in \mathbb{R}$ is possible. What steering angles are possible? The interval $[-\pi/2, \pi/2]$ is sufficiently large for the steering angle u_ϕ because any other value is equivalent to one between $-\pi/2$ and $\pi/2$. Steering angles of $\pi/2$ and $-\pi/2$ are problematic. To derive the expressions for $\dot{x}$ and $\dot{y}$, it was assumed that the car moves in the direction that the rear wheels are pointing. Imagine you are sitting on a tricycle and turn the front wheel perpendicular to the rear wheels (assigning $u_\phi = \pi/2$). If you are able to pedal, then the tricycle should rotate in place. This means that $\dot{x} = \dot{y} = 0$ because the center of the rear axle does not translate.

This strange behavior is not allowed for a standard automobile. A car with rear-wheel drive would probably skid the front wheels across the pavement. If a car with front-wheel drive attempted this, it should behave as a tricycle; however,

this is usually not possible because the front wheels would collide with the front axle when turned to $\phi = \pi/2$. Therefore, the simple car should have a maximum steering angle, $\phi_{max} < \pi/2$, and we require that $|\phi| \leq \phi_{max}$. Observe from Figure 13.1 that a maximum steering angle implies a *minimum turning radius*, ρ_{min}. For the case of a tricycle, $\rho_{min} = 0$. You may have encountered the problem of a minimum turning radius while trying to make an illegal U-turn. It is sometimes difficult to turn a car around without driving it off of the road.

Now return to the speed u_s. On level pavement, a real vehicle has a top speed, and its behavior should change dramatically depending on the speed. For example, if you want to drive along the minimum turning radius, you should not drive at 140km/hr. It seems that the maximum steering angle should reduce at higher speeds. This enters the realm of dynamics, which will be allowed after phase spaces are introduced in Section 13.2. Following this, some models of cars with dynamics will be covered in Sections 13.2.4 and 13.3.3.

It has been assumed implicitly that the simple car is moving slowly to safely neglect dynamics. A bound such as $|u_s| \leq 1$ can be placed on the speed without affecting the configurations that it can reach. The speed can even be constrained as $u_s \in \{-1, 0, 1\}$ without destroying reachability. Be careful, however, about a bound such as $0 \leq u_s \leq 1$. In this case, the car cannot drive in reverse! This clearly affects the set of reachable configurations. Imagine a car that is facing a wall and is unable to move in reverse. It may be forced to hit the wall as it moves.

Based on these considerations regarding the speed and steering angle, several interesting variations are possible:

Tricycle: $U = [-1, 1] \times [-\pi/2, \pi/2]$. Assuming front-wheel drive, the "car" can rotate in place if $u_\phi = \pi/2$ or $u_\phi = \pi/2$. This is unrealistic for a simple car. The resulting model is similar to that of the simple unicycle, which appears later in (13.18).

Simple Car [596]: $U = [-1, 1] \times (-\phi_{max}, \phi_{max})$. By requiring that $|u_\phi| \leq \phi_{max} < \pi/2$, a car with minimum turning radius $\rho_{min} = L/\tan\phi_{max}$ is obtained.

Reeds-Shepp Car [814, 923]: Further restrict the speed of the simple car so that $u_s \in \{-1, 0, 1\}$.[3] This model intuitively makes u_s correspond to three discrete "gears": reverse, park, or forward. An interesting question under this model is: What is the shortest possible path (traversed in $\mathbb{R}^2$ by the center of the rear axle) between two configurations in the absence of obstacles? This is answered in Section 15.3.

Dubins Car [294]: Remove the reverse speed $u_s = -1$ from the Reeds-Shepp car to obtain $u_s \in \{0, 1\}$ as the only possible speeds. The shortest paths in $\mathbb{R}^2$ for this car are quite different than for the Reeds-Shepp car; see Section 15.3.

[3] In many works, the speed $u_s = 0$ is not included. It appears here so that a proper termination condition can be defined.

The car that was shown in Figure 1.12a of Section 1.2 is even more restricted than the Dubins car because it is additionally forced to turn left.

Basic controllability issues have been studied thoroughly for the simple car. These will be covered in Section 15.4, but it is helpful to develop intuitive notions here to assist in understanding the planning algorithms of Chapter 14. The simple car is considered *nonholonomic* because there are differential constraints that cannot be completely integrated. This means that the car configurations are not restricted to a lower dimensional subspace of $\mathcal{C}$. The Reeds-Shepp car can be maneuvered into an arbitrarily small parking space, provided that a small amount of clearance exists. This property is called *small-time local controllability* and is presented in Section 15.1.3. The Dubins car is nonholonomic, but it does not possess this property. Imagine the difficulty of parallel parking without using the reverse gear. In an infinitely large parking lot without obstacles, however, the Dubins car can reach any configuration.

13.1.2.2 A differential drive

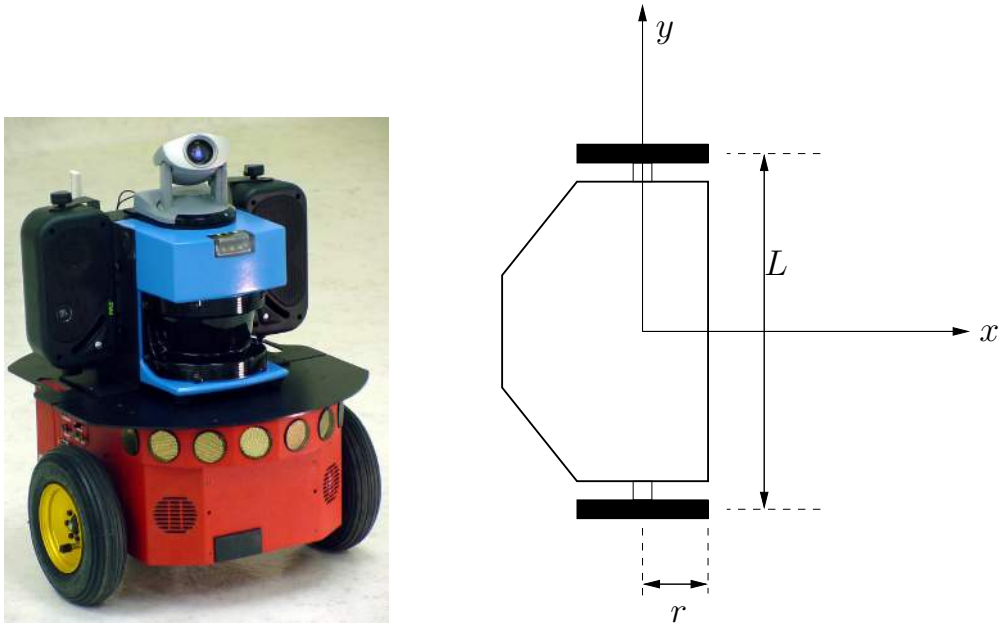
Most indoor mobile robots do not move like a car. For example, consider the mobile robotics platform shown in Figure 13.2a. This is an example of the most popular way to drive indoor mobile robots. There are two main wheels, each of which is attached to its own motor. A third wheel (not visible in Figure 13.2a) is placed in the rear to passively roll along while preventing the robot from falling over.

To construct a simple model of the constraints that arise from the differential drive, only the distance L between the two wheels, and the wheel radius, r, are necessary. See Figure 13.2b. The action vector $u = (u_r, u_l)$ directly specifies the two angular wheel velocities (e.g., in radians per second). Consider how the robot moves as different actions are applied. See Figure 13.3. If $u_l = u_r > 0$, then the robot moves forward in the direction that the wheels are pointing. The speed is proportional to r. In general, if $u_l = u_r$, then the distance traveled over a duration t of time is rtu_l (because tu_l is the total angular displacement of the wheels). If $u_l = -u_r \neq 0$, then the robot rotates clockwise because the wheels are turning in opposite directions. This motivates the placement of the body-frame origin at the center of the axle between the wheels. By this assignment, no translation occurs if the wheels rotate at the same rate but in opposite directions.

Based on these observations, the configuration transition equation is

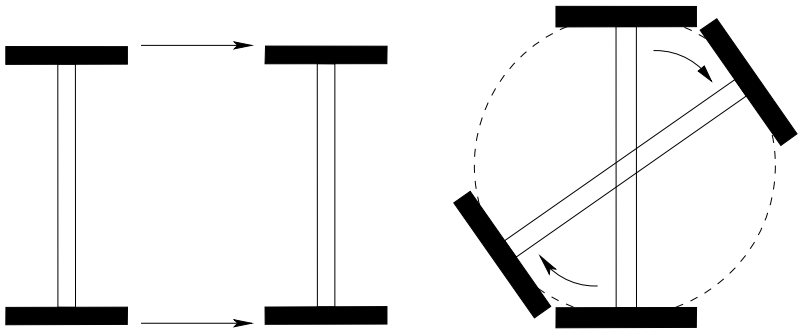
$$\begin{aligned} \dot{x} &= \frac{r}{2}(u_l + u_r)\cos\theta \\ \dot{y} &= \frac{r}{2}(u_l + u_r)\sin\theta \\ \dot{\theta} &= \frac{r}{L}(u_r - u_l). \end{aligned} \tag{13.16}$$

The translational part contains $\cos\theta$ and $\sin\theta$ parts, just like the simple car because the differential drive moves in the direction that its drive wheels are pointing.



(a) (b)

Figure 13.2: (a) The Pioneer 3-DX8 (courtesy of ActivMedia Robotics: MobileRobots.com), and many other mobile robots use a differential drive. In addition to the two drive wheels, a caster wheel (as on the bottom of an office chair) is placed in the rear center to prevent the robot from toppling over. (b) The parameters of a generic differential-drive robot.



(a) (b)

Figure 13.3: (a) Pure translation occurs when both wheels move at the same angular velocity; (b) pure rotation occurs when the wheels move at opposite velocities.

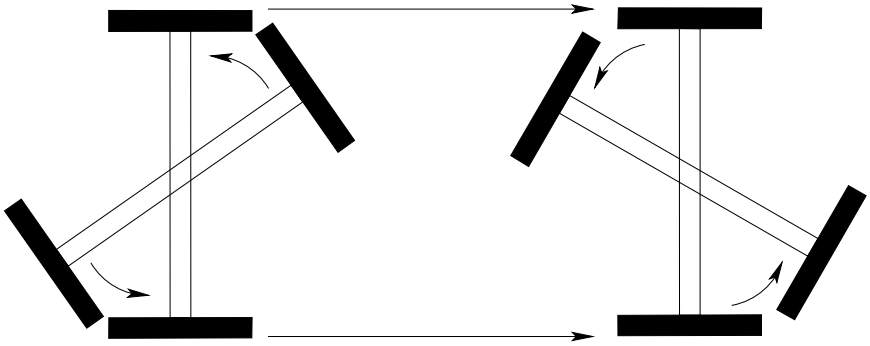


Figure 13.4: The shortest path traversed by the center of the axle is simply the line segment that connects the initial and goal positions in the plane. Rotations appear to be cost-free.

The translation speed depends on the average of the angular wheel velocities. To see this, consider the case in which one wheel is fixed and the other rotates. This initially causes the robot to translate at 1/2 of the speed in comparison to both wheels rotating. The rotational speed $\dot{\theta}$ is proportional to the change in angular wheel speeds. The robot's rotation rate grows linearly with the wheel radius but reduces linearly with respect to the distance between the wheels.

It is sometimes preferable to transform the action space. Let $u_\omega = (u_r + u_l)/2$ and $u_\psi = u_r - u_l$. In this case, u_ω can be interpreted as an action variable that means "translate," and u_ψ means "rotate." Using these actions, the configuration transition equation becomes

$$\begin{aligned} \dot{x} &= r u_\omega \cos\theta \\ \dot{y} &= r u_\omega \sin\theta \\ \dot{\theta} &= \frac{r}{L} u_\psi. \end{aligned} \tag{13.17}$$

In this form, the configuration transition equation resembles (13.15) for the simple car (try setting $u_\psi = \tan u_\phi$ and $u_s = r u_\omega$). A differential drive can easily simulate the motions of the simple car. For the differential drive, the rotation rate can be set independently of the translational velocity. The simple car, however, has the speed u_s appearing in the $\dot{\theta}$ expression. Therefore, the rotation rate depends on the translational velocity.

Recall the question asked about shortest paths for the Reeds-Shepp and Dubins cars. The same question for the differential drive turns out to be uninteresting because the differential drive can cause the center of its axle to follow any continuous path in $\mathbb{R}^2$. As depicted in Figure 13.4, it can move between any two configurations by: 1) first rotating itself to point the wheels to the goal position, which causes no translation; 2) translating itself to the goal position; and 3) rotating itself to the desired orientation, which again causes no translation. The total distance traveled by the center of the axle is always the Euclidean distance in $\mathbb{R}^2$ between the two desired positions.

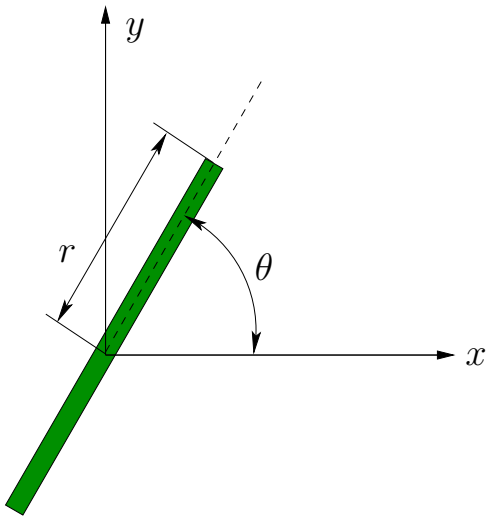


Figure 13.5: Viewed from above, the unicycle model has an action u_ω that changes the wheel orientation θ.

This may seem like a strange effect due to the placement of the coordinate origin. Rotations seem to have no cost. This can be fixed by optimizing the total amount of wheel rotation or time required, if the speed is held fixed [64]. Suppose that $u_r, u_l \in \{-1, 0, 1\}$. Determining the minimum time required to travel between two configurations is quite interesting and is covered in Section 15.3. This properly takes into account the cost of rotating the robot, even if it does not cause a translation.

13.1.2.3 A simple unicycle

Consider the simple model of a unicycle, which is shown in Figure 13.5. Ignoring balancing concerns, there are two action variables. The rider of the unicycle can set the pedaling speed and the orientation of the wheel with respect to the z-axis. Let σ denote the pedaling angular velocity, and let r be the wheel radius. The speed of the unicycle is $s = r\sigma$. In this model, the speed is set directly by an action variable u_s (alternatively, the pedaling rate could be an action variable u_σ, and the speed is derived as $s = ru_\sigma$). Let ω be the angular velocity of the unicycle orientation in the xy plane (hence, $\omega = \dot{\theta}$). Let ω be directly set by an action variable u_ω. The configuration transition equation is

$$\begin{aligned} \dot{x} &= u_s \cos\theta \\ \dot{y} &= u_s \sin\theta \\ \dot{\theta} &= u_\omega. \end{aligned} \tag{13.18}$$

This is just the differential drive equation (13.17) with $L = 1$ and the substitution $u_s = ru_\sigma$. Thus, a differential drive can simulate a unicycle. This may seem strange; however, it is possible because these models do not consider dynamics.

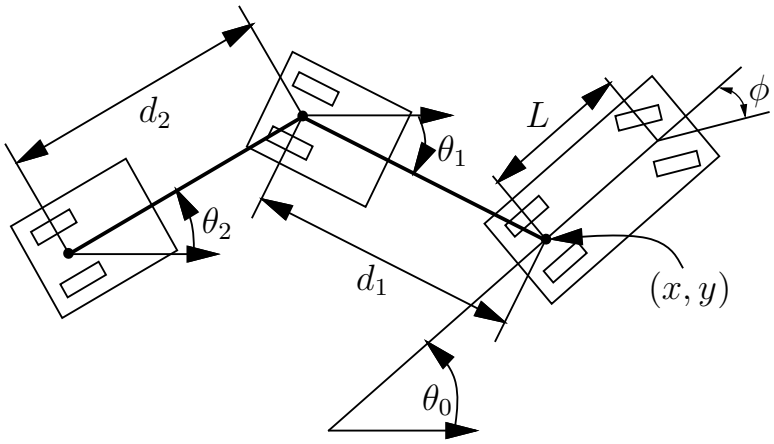


Figure 13.6: The parameters for a car pulling trailers.

Note that the unicycle can also simulate the simple-car model. Therefore, the tricycle and unicycle models are similar.

13.1.2.4 A car pulling trailers

An interesting extension of the simple car can be made by attaching one or more trailers. You may have seen a train of luggage carts on the tarmac at airports. There are many subtle issues for modeling the constraints for these models. The form of equations is very sensitive to the precise point at which the trailer is attached and also on the choice of body frames. One possibility for expressing the kinematics is to use the expressions in Section 3.3; however, these may lead to complications when analyzing the constraints. It is somewhat of an art to find a simple expression of the constraints. The model given here is adapted from [727].[4]

Consider a simple car that pulls k trailers as shown in Figure 13.6. Each trailer is attached to the center of the rear axle of the body in front of it. The important new parameter is the *hitch length* d_i which is the distance from the center of the rear axle of trailer i to the point at which the trailer is hitched to the next body. Using concepts from Section 3.3.1, the car itself contributes $\mathbb{R}^2 \times \mathbb{S}^1$ to $\mathcal{C}$, and each trailer contributes an $\mathbb{S}^1$ component to $\mathcal{C}$. The dimension of $\mathcal{C}$ is therefore $k + 3$. Let θ_i denote the orientation of the ith trailer, expressed with respect to the world frame.

[4]The original model required a continuous steering angle.

The configuration transition equation is

$$\begin{aligned}
\dot{x} &= s \cos\theta_0 \\
\dot{y} &= s \sin\theta_0 \\
\dot{\theta}_0 &= \frac{s}{L} \tan\phi \\
\dot{\theta}_1 &= \frac{s}{d_1} \sin(\theta_0 - \theta_1) \\
&\vdots \\
\dot{\theta}_i &= \frac{s}{d_i} \left(\prod_{j=1}^{i-1} \cos(\theta_{j-1} - \theta_j) \right) \sin(\theta_{i-1} - \theta_i) \\
&\vdots \\
\dot{\theta}_k &= \frac{s}{d_k} \left(\prod_{j=1}^{k-1} \cos(\theta_{j-1} - \theta_j) \right) \sin(\theta_{k-1} - \theta_k).
\end{aligned} \tag{13.19}$$

An interesting variation of this model is to allow the trailer wheels to be steered. For a single trailer, this leads to a model that resembles a *firetruck* [163].

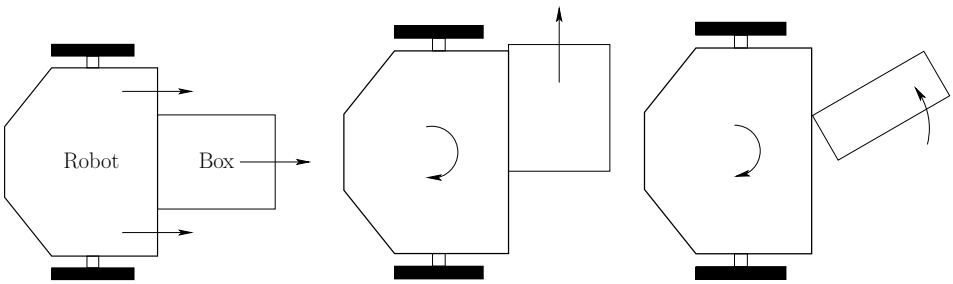
13.1.3 Other Examples of Velocity Constraints

The differential models seen so far were obtained from wheels that roll along a planar surface. Many generalizations are possible by considering other ways in which bodies can contact each other. In robotics, many interesting differential models arise in the context of manipulation. This section briefly covers some other examples of velocity constraints on the C-space. Once again, dynamics is neglected for now. Such models are sometimes classified as *quasi-static* because even though motions occur, some aspects of the model treat the bodies as if they were static. Such models are often realistic when moving at slow enough speeds.

13.1.3.1 Pushing a box

Imagine using a differential drive robot to push a box around on the floor, as shown in Figure 13.7a. It is assumed that the box is a convex polygon, one edge of which contacts the front of the robot. There are frictional contacts between the box and floor and also between the box and robot. Suppose that the robot is moving slowly enough so that dynamics are insignificant. It is assumed that the box cannot move unless the robot is moving. This prohibits manipulations such as "kicking" the box across the room. The term *stable pushing* [12, 671, 681] refers to the case in which the robot moves the box as if the box were rigidly attached to the robot.

As the robot pushes the box, the box may slide or rotate, as shown in Figures 13.7b and 13.7c, respectively. These cases are considered illegal because they do



(a) Stable pushing (b) Illegal sliding (c) Illegal rotation

Figure 13.7: Lynch and Mason showed that pushing a box is very much like driving the simple car: (a) With careful motions, the box will act as if it is attached to the robot. b) If it turns too sharply, however, the box will slide away; this induces limits on the steering angle. c) The box may alternatively rotate from sharp turns [671].

not constitute stable pushing. What motions of the robot are possible? Begin with the configuration transition equation of the differential drive robot, and then determine which constraints need to be placed on U to maintain stable pushing. Suppose that (13.17) is used. It is clear that only forward motion is possible because the robot immediately breaks contact with the box if the robot moves in the opposite direction. Thus, s must be positive (also, to fit the quasi-static model, s should be small enough so that dynamical effects become insignificant). How should the rotation rate ψ be constrained? Constraints on ψ depend on the friction model (e.g., Coulomb), the shape of the box, and the particular edge that is being pushed. Details on these constraints are given in [671, 681]. This leads to an interval $[a, b] \subseteq [-\pi/2, \pi/2]$, in which $a < 0$ and $b > 0$, and it is required that $\psi \in [a, b]$. This combination of constraints produces a motion model that is similar to the Dubins car. The main difference is that the maximum steering angle in the left and right directions may be different.

To apply this model for planning, it seems that the C-space should be $\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R}^2 \times \mathbb{S}^1$ because there are two rigid bodies. The manipulation planning framework of Section 7.3.2 can be applied to obtain a hybrid system and manipulation graph that expresses the various ways in which the robot can contact the box or fail to contact the box. For example, the robot may be able to push the box along one of several possible edges. If the robot becomes stuck, it can change the pushing edge to move the box in a new direction.

13.1.3.2 Flying an airplane

The Dubins car model from Section 13.1.2 can be extended to 3D worlds to provide a simple aircraft flight model that may be reasonable for air traffic analysis. First suppose that the aircraft maintains a fixed altitude and is capable only of yaw rotations. In this case, (13.15) could be used directly by imposing the constraint

that $s = 1$ (or some suitable positive speed). This is equivalent to the Dubins car, except that $s = 0$ is prohibited because it would imply that the aircraft can instantaneously stop in the air. This model assumes that the aircraft is small relative to the C-space. A more precise model should take into account pitch and roll rotations, disturbances, and dynamic effects. These would become important, for example, in studying the flight stability of an aircraft design. Such concerns are neglected here.

Now consider an aircraft that can change its altitude, in addition to executing motions like the Dubins car. In this case let $\mathcal{C} = \mathbb{R}^3 \times \mathbb{S}^1$, in which the extra $\mathbb{R}$ represents the altitude with respect to flying over a flat surface. A configuration is represented as $q = (x, y, z, \theta)$. Let u_z denote an action that directly causes a change in the altitude: $\dot{z} = u_z$. The steering action u_ϕ is the same as in the Dubins car model. The configuration transition equation is

$$\begin{aligned} \dot{x} &= \cos\theta & \dot{z} &= u_z \\ \dot{y} &= \sin\theta & \dot{\theta} &= u_\omega. \end{aligned} \tag{13.20}$$

For a fixed value of $u = (u_z, u_\omega)$ such that $u_z \neq 0$ and $u_\omega \neq 0$, a helical path results. The central axis of the helix is parallel to the z-axis, and projection of the path down to the xy plane is a circle or circular arc. Maximum absolute values should be set for u_z and u_ω based on the maximum possible altitude and yaw rate changes of the aircraft.

13.1.3.3 Rolling a ball

Instead of a wheel, consider rolling a ball in the plane. Place a ball on a table and try rolling it with your palm placed flat on top of it. It should feel like there are two degrees of freedom: rolling forward and rolling side to side. The ball should not be able to spin in place. The directions can be considered as two action variables. The total degrees of freedom of the ball is five, however, because it can achieve any orientation in $SO(3)$ and any (x, y) position in the plane; thus, $\mathcal{C} = \mathbb{R}^2 \times SO(3)$. Given that there are only two action variables, is it possible to roll the ball into any configuration? It is shown in [632, 491] that this is possible, even for the more general problem of one sphere rolling on another (the plane is a special case of a sphere with infinite radius). This problem can actually arise in robotic manipulation when a spherical object come into contact (e.g., a robot hand may have fingers with spherical tips); see [103, 676, 725, 729].

The resulting transition equation was shown in [716] (also see [725]) to be

$$\begin{aligned} \dot{\theta} &= -u_2 \\ \dot{\phi} &= \frac{u_1}{\cos\theta} \\ \dot{x} &= -u_1 \rho \sin\psi - u_2 \rho \cos\psi \\ \dot{y} &= -u_1 \rho \cos\psi + u_2 \rho \sin\psi \\ \dot{\psi} &= -u_1 \tan\theta. \end{aligned} \tag{13.21}$$

In these equations, x and y are the position on the contact point in the plane, and θ and ϕ are the position of the contact point in the ball frame and are expressed using spherical coordinates. The radius of the ball is ρ. Finally, ψ expresses the orientation of the ball with respect to the contact point.

13.1.3.4 Trapped on a surface

It is possible that the constraints cause the configuration to be trapped on a lower dimensional surface. Let $\mathcal{C} = \mathbb{R}^2$, and consider the system

$$\dot{x} = yu \qquad\qquad \dot{y} = -xu, \tag{13.22}$$

for $(x, y) \in \mathbb{R}^2$ and $u \in U = \mathbb{R}$. What are the integral curves for a constant action $u \neq 0$? From any point $(x, y) \in \mathbb{R}^2$, the trajectory follows a circle of radius $\sqrt{x^2 + y^2}$ centered at the origin. The speed along the circle is determined by $|u|$, and the direction is determined by the sign of u. Therefore, (13.22) indicates that the configuration is confined to a circle. Other than that, there are no further constraints.

Suppose that the initial configuration is given as (x_0, y_0). Since the configuration is confined to a circle, the C-space could alternatively be defined as $\mathcal{C} = \mathbb{S}^1$. Each point on $\mathbb{S}^1$ can be mapped to the circle that has radius $r = \sqrt{x_0^2 + y_0^2}$ and center at $(0, 0)$. In this case, there are no differential constraints on the velocities, provided that motions are trapped on the circle. Any velocity in the one-dimensional tangent space at points on the circle is allowed. This model is equivalent to (13.22).

Now consider the possible trajectories that are constrained to traverse a circle,

$$h(x, y) = x^2 + y^2 - r^2 = 0. \tag{13.23}$$

This means that for all time t,

$$h(x(t), y(t)) = x(t)^2 + y(t)^2 - r^2 = 0. \tag{13.24}$$

To derive a constraint on velocities, take the derivative with respect to time, which yields

$$\frac{dh(x, y)}{dt} = 2x\dot{x} + 2y\dot{y} = 0. \tag{13.25}$$

This is an example of a Pfaffian constraint, as given in (13.5). The parametric form of this differential constraint happens to be (13.22). Any velocity vector that is a multiple of $(y, -x)$ satisfies (13.25). When expressed as a differential constraint, the radius r does not matter. This is because it is determined from the initial configuration.

What just occurred here is a special case of a *completely integrable* differential model. In general, if the model $\dot{q} = f(q, u)$ can be expressed as the time derivative of constraints of the form $h(q) = 0$, then the configuration transition equation is said to be *completely integrable*. Obtaining an implicit differential model from

constraints of the form $h_i(q) = 0$ is not difficult. Each constraint is differentiated to obtain

$$\frac{dh_i(q)}{dt} = 0. \tag{13.26}$$

For example, such constraints arise from closed kinematic chains, as in Section 4.4, and the implicit differential model just expresses the condition that velocities must lie in the tangent space to the constraints. It may be difficult, however, to obtain a parametric form of the differential model. Possible velocity vectors can be computed at any particular q, however, by using the linear algebra techniques described in Section 7.4.1.

It is even quite difficult to determine whether a differential model is completely integrable, which means that the configurations are trapped on a lower dimensional surface. For some systems, to be described by (13.41), this will be solved by the Frobenius Theorem in 15.4.2. If such systems are not completely integrable, they are called *nonholonomic*; otherwise, they are called *holonomic*. In general, even if a model is theoretically integrable, actually performing the integration is another issue. In most cases, it is difficult or impossible to integrate the model.

Therefore, it is sometimes important to work directly with constraints in differential form, even if they are integrable. Furthermore, methods for planning under differential constraints can be applied to problems that have constraints of the form $h(q) = 0$. This, for example, implies that motion planning for closed kinematic chains can be performed by planning algorithms designed to handle differential constraints.

13.2 Phase Space Representation of Dynamical Systems

The differential constraints defined in Section 13.1 are often called *kinematic* because they can be expressed in terms of velocities on the C-space. This formulation is useful for many problems, such as modeling the possible directions of motions for a wheeled mobile robot. It does not, however, enable dynamics to be expressed. For example, suppose that the simple car is traveling quickly. Taking dynamics into account, it should not be able to instantaneously start and stop. For example, if it is heading straight for a wall at full speed, any reasonable model should not allow it to apply its brakes from only one millimeter away and expect it to avoid collision. Due to momentum, the required stopping distance depends on the speed. You may have learned this from a drivers education course.

To account for momentum and other aspects of dynamics, higher order differential equations are needed. There are usually constraints on acceleration $\ddot{q}$, which is defined as $d\dot{q}/dt$. For example, the car may only be able to decelerate at some maximum rate without skidding the wheels (or tumbling the vehicle). Most often, the actions are even expressed in terms of higher order derivatives. For example, the floor pedal of a car may directly set the acceleration. It may be reasonable

to consider the amount that the pedal is pressed as an action variable. In this case, the configuration must be obtained by two integrations. The first yields the velocity, and the second yields the configuration.

The models for dynamics therefore involve acceleration $\ddot{q}$ in addition to velocity $\dot{q}$ and configuration q. Once again, both implicit and parametric models exist. For an implicit model, the constraints are expressed as

$$g_i(\ddot{q}, \dot{q}, q) = 0. \tag{13.27}$$

For a parametric model, they are expressed as

$$\ddot{q} = f(\dot{q}, q, u). \tag{13.28}$$

13.2.1 Reducing Degree by Increasing Dimension

Taking into account constraints on higher order derivatives seems substantially more complicated. This section explains a convenient trick that converts constraints that have higher order derivatives into a new set of constraints that has only first-order derivatives. This involves the introduction of a *phase space*, which has more dimensions than the original C-space. Thus, there is a trade-off because the dimension is increased; however, it is widely accepted that increasing the dimension of the space is often easier than dealing with higher order derivatives. In general, the term *state space* will refer to either C-spaces or phase spaces derived from them.

13.2.1.1 The scalar case

To make the discussion concrete, consider the following differential equation:

$$\ddot{y} - 3\dot{y} + y = 0, \tag{13.29}$$

in which y is a scalar variable, $y \in \mathbb{R}$. This is a second-order differential equation because of $\ddot{y}$. A *phase space* can be defined as follows. Let $x = (x_1, x_2)$ denote a two-dimensional *phase vector*, which is defined by assigning $x_1 = y$ and $x_2 = \dot{y}$. The terms state space and state vector will be used interchangeably with phase space and phase vector, respectively, in contexts in which the phase space is defined. Substituting the equations into (13.29) yields

$$\ddot{y} - 3x_2 + x_1 = 0. \tag{13.30}$$

So far, this does not seem to have helped. However, $\ddot{y}$ can be expressed as either $\dot{x}_2$ or $\ddot{x}_1$. The first choice is better because it is a lower order derivative. Using $\dot{x}_2 = \ddot{y}$, the differential equation becomes

$$\dot{x}_2 - 3x_2 + x_1 = 0. \tag{13.31}$$

Is this expression equivalent to (13.29)? By itself it is not. There is one more constraint, $x_2 = \dot{x}_1$. In implicit form, $\dot{x}_1 - x_2 = 0$. The key to making the

phase space approach work correctly is to relate some of the phase variables by derivatives.

Using the phase space, we just converted the second-order differential equation (13.29) into two first-order differential equations,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 3x_2 - x_1, \end{aligned} \tag{13.32}$$

which are obtained by solving for $\dot{x}_1$ and $\dot{x}_2$. Note that (13.32) can be expressed as $\dot{x} = f(x)$, in which f is a function that maps from $\mathbb{R}^2$ into $\mathbb{R}^2$.

The same approach can be used for any differential equation in implicit form, $g(\ddot{y}, \dot{y}, y) = 0$. Let $x_1 = y$, $x_2 = \dot{y}$, and $\dot{x}_2 = \ddot{y}$. This results in the implicit equations $g(\dot{x}_2, x_2, x_1) = 0$ and $\dot{x}_1 = x_2$. Now suppose that there is a scalar action $u \in U = \mathbb{R}$ represented in the differential equations. Once again, the same approach applies. In implicit form, $g(\ddot{y}, \dot{y}, y, u) = 0$ can be expressed as $g(\dot{x}_2, x_2, x_1, u) = 0$.

Suppose that a given acceleration constraint is expressed in parametric form as $\ddot{y} = h(\dot{y}, y, u)$. This often occurs in the dynamics models of Section 13.3. This can be converted into a *phase transition equation* or *state transition equation* of the form $\dot{x} = f(x, u)$, in which $f : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2$. The expression is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= h(x_2, x_1, u). \end{aligned} \tag{13.33}$$

For a second-order differential equation, two initial conditions are usually given. The values of $y(0)$ and $\dot{y}(0)$ are needed to determine the exact position $y(t)$ for any $t \geq 0$. Using the phase space representation, no higher order initial conditions are needed because any point in phase space indicates both y and $\dot{y}$. Thus, given an initial point in the phase and $u(t)$ for all $t \geq 0$, $y(t)$ can be determined.

Example 13.3 (Double Integrator) The *double integrator* is a simple yet important example that nicely illustrates the phase space. Suppose that a second-order differential equation is given as $\ddot{q} = u$, in which q and u are chosen from $\mathbb{R}$. In words, this means that the action directly specifies acceleration. Integrating[5] once yields the velocity $\dot{q}$ and performing a double integration yields the position q. If $q(0)$ and $\dot{q}(0)$ are given, and $u(t')$ is specified for all $t' \in [0, t)$, then $\dot{q}(t)$ and $q(t)$ can be determined for any $t > 0$.

A two-dimensional phase space $X = \mathbb{R}^2$ is defined in which

$$x = (x_1, x_2) = (q, \dot{q}). \tag{13.34}$$

The state (or phase) transition equation $\dot{x} = f(x, u)$ is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u. \end{aligned} \tag{13.35}$$

[5]Wherever integrals are performed, it will be assumed that the integrands are integrable.

To determine the state trajectory, initial values $x_1(0) = q_0$ (position) and $x_2(0) = \dot{q}_0$ (velocity) must be given in addition to the action history. If u is constant, then the state trajectory is quadratic because it is obtained by two integrations of a constant function. ■

13.2.1.2 The vector case

The transformation to the phase space can be extended to differential equations in which there are time derivatives in more than one variable. Suppose that q represents a configuration, expressed using a coordinate neighborhood on a smooth n-dimensional manifold $\mathcal{C}$. Second-order constraints of the form $g(\ddot{q}, \dot{q}, q) = 0$ or $g(\ddot{q}, \dot{q}, q, u) = 0$ can be expressed as first-order constraints in a $2n$-dimensional state space. Let x denote the $2n$-dimensional phase vector. By extending the method that was applied to the scalar case, x is defined as $x = (q, \dot{q})$. For each integer i such that $1 \leq i \leq n$, $x_i = q_i$. For each i such that $n + 1 \leq i \leq 2n$, $x_i = \dot{q}_{i-n}$. These substitutions can be made directly into an implicit constraint to reduce the order to one.

Suppose that a set of n differential equations is expressed in parametric form as $\ddot{q} = h(q, \dot{q}, u)$. In the phase space, there are $2n$ differential equations. The first n correspond to the phase space definition $\dot{x}_i = x_{n+i}$, for each i such that $1 \leq i \leq n$. These hold because $x_{n+i} = \dot{q}_i$ and $\dot{x}_i$ is the time derivative of $\dot{q}_i$ for $i \leq n$. The remaining n components of $\dot{x} = f(x, u)$ follow directly from h by substituting the first n components of x in the place of q and the remaining n in the place of $\dot{q}$ in the expression $h(q, \dot{q}, u)$. The result can be denoted as $h(x, u)$ (obtained directly from $h(q, \dot{q}, u)$). This yields the final n equations as $\dot{x}_i = h_{i-n}(x, u)$, for each i such that $n + 1 \leq i \leq 2n$. These $2n$ equations define a *phase (or state) transition equation* of the form $\dot{x} = f(x, u)$. Now it is clear that constraints on acceleration can be manipulated into velocity constraints on the phase space. This enables the tangent space concepts from Section 8.3 to express constraints that involve acceleration. Furthermore, the state space X is the tangent bundle (defined in (8.9) for $\mathbb{R}^n$ and later in (15.67) for any smooth manifold) of $\mathcal{C}$ because q and $\dot{q}$ together indicate a tangent space $T_q(\mathcal{C})$ and a particular tangent vector $\dot{q} \in T_q(\mathcal{C})$.

13.2.1.3 Higher order differential constraints

The phase space idea can even be applied to differential equations with order higher than two. For example, a constraint may involve the time derivative of acceleration $q^{(3)}$, which is often called *jerk*. If the differential equations involve jerk variables, then a $3n$-dimensional phase space can be defined to obtain first-order constraints. In this case, each q_i, $\dot{q}_i$, and $\ddot{q}_i$ in a constraint such as $g(q^{(3)}, \ddot{q}, \dot{q}, q, u) = 0$ is defined as a phase variable. Similarly, kth-order differential constraints can be reduced to first-order constraints by introducing a kn-dimensional phase space.

Example 13.4 (Chain of Integrators) A simple example of higher order differential constraints is the *chain of integrators*.[6] This is a higher order generalization of Example 13.3. Suppose that a kth-order differential equation is given as $q^{(k)} = u$, in which q and u are scalars, and $q^{(k)}$ denotes the kth derivative of q with respect to time.

A k-dimensional phase space X is defined in which

$$x = (q, \dot{q}, \ddot{q}, q^{(3)}, \ldots, q^{(k-1)}). \tag{13.36}$$

The state (or phase) transition equation $\dot{x} = f(x, u)$ is $\dot{x}_i = x_{i+1}$ for each i such that $1 \leq i \leq n-1$, and $\dot{x}_n = u$. Together, these n individual equations are equivalent to $q^{(k)} = u$.

The initial state specifies the initial position and all time derivatives up to order $k-1$. Using these and the action u, the state trajectory can be obtained by a chain of integrations. ■

You might be wondering whether derivatives can be eliminated completely by introducing a phase space that has high enough dimension. This does actually work. For example, if there are second-order constraints, then a $3n$-dimensional phase space can be introduced in which $x = (q, \dot{q}, \ddot{q})$. This enables constraints such as $g(q, \dot{q}, \ddot{q}) = 0$ to appear as $g(x) = 0$. The trouble with using such formulations is that the state must follow the constraint surface in a way that is similar to traversing the solution set of a closed kinematic chain, as considered in Section 4.4. This is why tangent spaces arose in that context. In either case, the set of allowable velocities becomes constrained at every point in the space.

Problems defined using phase spaces typically have an interesting property known as *drift*. This means that for some $x \in X$, there does *not* exist any $u \in U$ such that $f(x, u) = 0$. For the examples in Section 13.1.2, such an action always existed. These were examples of *driftless systems*. This was possible because the constraints did not involve dynamics. In a dynamical system, it is impossible to instantaneously stop due to momentum, which is a form of drift. For example, a car will "drift" into a brick wall if it is 3 meters way and traveling 100 km/hr in the direction of the wall. There exists no action (e.g., stepping firmly on the brakes) that could instantaneously stop the car. In general, there is no way to instantaneously stop in the phase space.

13.2.2 Linear Systems

Now that the phase space has been defined as a special kind of state space that can handle dynamics, it is convenient to classify the kinds of differential models that can be defined based on their mathematical form. The class of *linear systems* has been most widely studied, particularly in the context of control theory. The

[6] It is called this because in block diagram representations of systems it is depicted as a chain of integrator blocks.

reason is that many powerful techniques from linear algebra can be applied to yield good control laws [192]. The ideas can also be generalized to linear systems that involve optimality criteria [28, 570], nature [95, 564], or multiple players [59].

Let $X = \mathbb{R}^n$ be a phase space, and let $U = \mathbb{R}^m$ be an action space for $m \leq n$. A *linear system* is a differential model for which the state transition equation can be expressed as

$$\dot{x} = f(x, u) = Ax + Bu, \tag{13.37}$$

in which A and B are constant, real-valued matrices of dimensions $n \times n$ and $n \times m$, respectively.

Example 13.5 (Linear System Example) For a simple example of (13.37), suppose $X = \mathbb{R}^3$, $U = \mathbb{R}^2$, and let

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{2} & 1 \\ 1 & -1 & 4 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{13.38}$$

Performing the matrix multiplications reveals that all three equations are linear in the state and action variables. Compare this to the discrete-time linear Gaussian system shown in Example 11.25. ■

Recall from Section 13.1.1 that k linear constraints *restrict* the velocity to an $(n - k)$-dimensional hyperplane. The linear model in (13.37) is in parametric form, which means that each action variable may *allow* an independent degree of freedom. In this case, $m = n - k$. In the extreme case of $m = 0$, there are no actions, which results in $\dot{x} = Ax$. The phase velocity $\dot{x}$ is fixed for every point $x \in X$. If $m = 1$, then at every $x \in X$ a one-dimensional set of velocities may be chosen using u. This implies that the direction is fixed, but the magnitude is chosen using u. In general, the set of allowable velocities at a point $x \in \mathbb{R}^n$ is an m-dimensional linear subspace of the tangent space $T_x(\mathbb{R}^n)$ (if B is nonsingular).

In spite of (13.37), it may still be possible to reach all of the state space from any initial state. It may be costly, however, to reach a nearby point because of the restriction on the tangent space; it is impossible to command a velocity in some directions. For the case of nonlinear systems, it is sometimes possible to quickly reach any point in a small neighborhood of a state, while remaining in a small region around the state. Such issues fall under the general topic of controllability, which will be covered in Sections 15.1.3 and 15.4.3.

Although not covered here, the *observability* of the system is an important topic in control [192, 478]. In terms of the I-space concepts of Chapter 11, this means that a sensor of the form $y = h(x)$ is defined, and the task is to determine the current state, given the history I-state. If the system is observable, this means that the nondeterministic I-state is a single point. Otherwise, the system may only be partially observable. In the case of linear systems, if the sensing model is also linear,

$$y = h(x) = Cy, \tag{13.39}$$

then simple matrix conditions can be used to determine whether the system is observable [192]. Nonlinear observability theory also exists [478].

As in the case of discrete planning problems, it is possible to define differential models that depend on time. In the discrete case, this involves a dependency on stages. For the continuous-stage case, a *time-varying linear system* is defined as

$$\dot{x} = f(x(t), u(t), t) = A(t)x(t) + B(t)u(t). \tag{13.40}$$

In this case, the matrix entries are allowed to be functions of time. Many powerful control techniques can be easily adapted to this case, but it will not be considered here because most planning problems are *time-invariant* (or stationary).

13.2.3 Nonlinear Systems

Although many powerful control laws can be developed for linear systems, the vast majority of systems that occur in the physical world fail to be linear. Any differential models that do not fit (13.37) or (13.40) are called *nonlinear systems*. All of the models given in Section 13.1.2 are nonlinear systems for the special case in which $X = \mathcal{C}$.

One important family of nonlinear systems actually appears to be linear in some sense. Let X be a smooth n-dimensional manifold, and let $U \subseteq \mathbb{R}^m$. Let $U = \mathbb{R}^m$ for some $m \leq n$. Using a coordinate neighborhood, a nonlinear system of the form

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i \tag{13.41}$$

for smooth functions f and g_i is called a *control-affine system* or *affine-in-control system*.[7] These have been studied extensively in nonlinear control theory [478, 846]. They are linear in the actions but nonlinear with respect to the state. See Section 15.4.1 for further reading on control-affine systems.

For a control-affine system it is not necessarily possible to obtain zero velocity because f causes drift. The important special case of a *driftless* control-affine system occurs if $f \equiv 0$. This is written as

$$\dot{x} = \sum_{i=1}^{m} g_i(x)u_i. \tag{13.42}$$

By setting $u_i = 0$ for each i from 1 to m, zero velocity, $\dot{x} = 0$, is obtained.

Example 13.6 (Nonholonomic Integrator) One of the simplest examples of a driftless control-affine system is the *nonholonomic integrator* introduced in control literature by Brockett in [142]. It some times referred to as *Brockett's system*, or the *Heisenberg system* because it arises in quantum mechanics [112]. Let $X =$

[7] Be careful not to confuse control-affine systems with *affine control systems*, which are of the form $\dot{x} = Ax + Bu + w$, for some constant matrices A, B and a constant vector w.

$\mathbb{R}^3$, and let the set of actions $U = \mathbb{R}^2$. The state transition equation for the nonholonomic integrator is

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1 u_2 - x_2 u_1. \end{aligned} \tag{13.43}$$

■

Many nonlinear systems can be expressed implicitly using Pfaffian constraints, which appeared in Section 13.1.1, and can be generalized from C-spaces to phase spaces. In terms of X, a Pfaffian constraint is expressed as

$$g_1(x)\dot{x}_1 + g_2(x)\dot{x}_2 + \cdots + g_n(x)\dot{x}_n = 0. \tag{13.44}$$

Even though the equation is linear in $\dot{x}$, a nonlinear dependency on x is allowed.

Both holonomic and nonholonomic models may exist for phase spaces, just as in the case of C-spaces in Section 13.1.3. The Frobenius Theorem, which is covered in Section 15.4.2, can be used to determine whether control-affine systems are completely integrable.

13.2.4 Extending Models by Adding Integrators

The differential models from Section 13.1 may seem unrealistic in many applications because actions are required to undergo instantaneous changes. For example, in the simple car, the steering angle and speed may be instantaneously changed to any value. This implies that the car is capable of instantaneous acceleration changes. This may be a reasonable approximation if the car is moving slowly (for example, to analyze parallel-parking maneuvers). The model is ridiculous, however, at high speeds.

Suppose a state transition equation of the form $\dot{x} = f(x, u)$ is given in which the dimension of X is n. The model can be enhanced as follows:

1. Select an action variable u_i.
2. Rename the action variable as a new state variable, $x_{n+1} = u_i$.
3. Define a new action variable u_i' that takes the place of u_i.
4. Extend the state transition equation by one dimension by introducing $\dot{x}_{n+1} = u_i'$.

This enhancement will be referred to as *placing an integrator in front of* u_i. This procedure can be applied incrementally as many times as desired, to create a chain of integrators from any action variable. It can also be applied to different action variables.

13.2.4.1 Better unicycle models

Improvements to the models in Section 13.1 can be made by placing integrators in front of action variables. For example, consider the unicycle model (13.18). Instead of directly setting the speed using u_s, suppose that the speed is obtained by integration of an action u_a that represents acceleration. The equation $\dot{s} = u_a$ is used instead of $s = u_s$, which means that the action sets the *change* in speed. If u_a is chosen from some bounded interval, then the speed is a continuous function of time.

How should the transition equation be represented in this case? The set of possible values for u_a imposes a second-order constraint on x and y because double integration is needed to determine their values. By applying the phase space idea, s can be considered as a phase variable. This results in a four-dimensional phase space, in which each state is (x, y, θ, s). The state (or phase) transition equation is

$$\begin{aligned} \dot{x} &= s\cos\theta & \dot{\theta} &= u_\omega \\ \dot{y} &= s\sin\theta & \dot{s} &= u_a, \end{aligned} \tag{13.45}$$

which should be compared to (13.18). The action u_s was replaced by s because now speed is a phase variable, and an extra equation was added to reflect the connection between speed and acceleration.

The integrator idea can be applied again to make the unicycle orientations a continuous function of time. Let u_α denote an angular acceleration action. Let ω denote the angular velocity, which is introduced as a new state variable. This results in a five-dimensional phase space and a model called the *second-order unicycle*:

$$\begin{aligned} \dot{x} &= s\cos\theta & \dot{s} &= u_a \\ \dot{y} &= s\sin\theta & \dot{\omega} &= u_\alpha \\ \dot{\theta} &= \omega, \end{aligned} \tag{13.46}$$

in which $u = (u_a, u_\alpha)$ is a two-dimensional action vector. In some contexts, s may be fixed at a constant value, which implies that u_a is fixed to $u_a = 0$.

13.2.4.2 A continuous-steering car

As another example, consider the simple car. As formulated in (13.15), the steering angle is allowed to change discontinuously. For simplicity, suppose that the speed is fixed at $s = 1$. To make the steering angle vary continuously over time, let u_ω be an action that represents the velocity of the steering angle: $\dot{\phi} = u_\omega$. The result is a four-dimensional state space, in which each state is represented as (x, y, θ, ϕ). This yields a *continuous-steering car*,

$$\begin{aligned} \dot{x} &= \cos\theta & \dot{\theta} &= \frac{\tan\phi}{L} \\ \dot{y} &= \sin\theta & \dot{\phi} &= u_\omega, \end{aligned} \tag{13.47}$$

in which there are two action variables, u_s and u_ω. This model was used for planning in [849].

A second integrator can be applied to make the steering angle a C^1 smooth function of time. Let ω be a state variable, and let u_α denote the angular acceleration of the steering angle. In this case, the state vector is $(x, y, \theta, \phi, \omega)$, and the state transition equation is

$$\begin{aligned} \dot{x} &= \cos\theta & \dot{\phi} &= \omega \\ \dot{y} &= \sin\theta & \dot{\omega} &= u_\alpha \\ \dot{\theta} &= \frac{\tan\phi}{L}. \end{aligned} \tag{13.48}$$

Integrators can be applied any number of times to make any variables as smooth as desired. Furthermore, the rate of change in each case can be bounded due to limits on the phase variables and on the action set.

13.2.4.3 Smooth differential drive

A *second-order differential drive* model can be made by defining actions u_l and u_r that accelerate the motors, instead of directly setting their velocities. Let ω_l and ω_r denote the left and right motor angular velocities, respectively. The resulting state transition equation is

$$\begin{aligned} \dot{x} &= \frac{r}{2}(\omega_l + \omega_r)\cos\theta & \dot{\omega}_l &= u_l \\ \dot{y} &= \frac{r}{2}(\omega_l + \omega_r)\sin\theta & \dot{\omega}_r &= u_r \\ \dot{\theta} &= \frac{r}{L}(\omega_r - \omega_l). \end{aligned} \tag{13.49}$$

In summary, an important technique for making existing models somewhat more realistic is to insert one or more integrators in front of any action variables. The dimension of the phase space increases with the introduction of each integrator. A single integrator forces an original action to become continuous over time. If the new action is bounded, then the rate of change of the original action is bounded in places where it is differentiable (it is Lipschitz in general, as expressed in (8.16)). Using a double integrator, the original action is forced to be C^1 smooth. Chaining more integrators on an action variable further constrains its values. In general, k integrators can be chained in front of an original action to force it to be C^{k-1} smooth and respect Lipschitz bounds.

One important limitation, however, is that to make realistic models, other variables may depend on the new phase variables. For example, if the simple car is traveling fast, then we should not be able to turn as sharply as in the case of a slow-moving car (think about how sharply you can turn the wheel while parallel parking in comparison to driving on the highway). The development of better differential models ultimately requires careful consideration of mechanics. This provides motivation for Sections 13.3 and 13.4.

13.3 Basic Newton-Euler Mechanics

Mechanics is a vast and difficult subject. It is virtually impossible to provide a thorough introduction in a couple of sections. Here, the purpose instead is to overview some of the main concepts and to provide some models that may be used with the planning algorithms in Chapter 14. The presentation in this section and in Section 13.4 should hopefully stimulate some further studies in mechanics (see the suggested literature at the end of the chapter). On the other hand, if you are only interested in *using* the differential models, then you can safely skip their derivations. Just keep in mind that all differential models produced in this section end with the form $\dot{x} = f(x, u)$, which is ready to use in planning algorithms.

There are two important points to keep in mind while studying mechanics:

1. The models are based on maintaining consistency with experimental observations about how bodies behave in the physical world. These observations depend on the kind of experiment. In a particular application, many effects may be insignificant or might not even be detectable by an experiment. For example, it is difficult to detect relativistic effects using a radar gun that measures automobile speed. It is therefore important to specify any simplifying assumptions regarding the world and the kind of experiments that will be performed in it.

2. The approach is usually to express some laws that translate into constraints on the allowable velocities in the phase space. This means that implicit representations are usually obtained in mechanics, and they must be converted into parametric form. Furthermore, most treatments of mechanics do not explicitly mention action variables; these arise from the intention of *controlling* the physical world. From the perspective of mechanics, the actions can be assumed to be already determined. Thus, constraints appear as $g(\dot{x}, x) = 0$, instead of $g(\dot{x}, x, u) = 0$.

Several formulations of mechanics arrive at the same differential constraints, but from different mathematical reasoning. The remainder of this chapter overviews three schools of thought, each of which is more elegant and modern than the one before. The easiest to understand is Newton-Euler mechanics, which follows from Newton's famous laws of physics and is covered in this section. Lagrangian mechanics is covered in Section 13.4.1 and arrives at the differential constraints using very general principles of optimization on a space of functions (i.e., calculus of variations). Hamiltonian mechanics, covered in Section 13.4.4, defines a higher dimensional state space on which the differential constraints can once again be obtained by optimization.

13.3.1 The Newtonian Model

The most basic formulation of mechanics goes back to Newton and Euler, and parts of it are commonly studied in basic physics courses. Consider a *world* $\mathcal{W}$

defined as in Section 3.1, except here a 1D world $\mathcal{W} = \mathbb{R}$ is allowed, in addition to 2D and 3D worlds. A notion of time is also needed. The space of motions that can be obtained in the space-time continuum can be formalized as a Galilean group [39]; however, the presentation here will utilize standard intuitive notions of time and Euclidean space. It is also assumed that any relativistic effects due to curvature of the time-space continuum are nonexistent (Newton and Euler did not know about this, and it is insignificant for most small-scale mechanical systems on or near the earth).

Inertial coordinate frames Central to Newton-Euler mechanics is the idea that points in $\mathcal{W}$ are expressed using an *inertial coordinate frame*. Imagine locating the origin and axes of $\mathcal{W}$ somewhere in our universe. They need to be fixed in a way that does not interfere with our observations of the basic laws of motion. Imagine that we are playing racquetball in an indoor court and want to model the motion of the ball as it bounces from wall to wall. If the coordinate frame is rigidly attached to the ball, it will appear that the ball never moves; however, the walls, earth, and the rest of the universe will appear to spin wildly around the ball (imagine we have camera that points along some axis of the ball frame – you could quickly become ill trying to follow the movie). If the coordinate frame is fixed with respect to the court, then sensible measurements of the ball positions would result (the movie would also be easier to watch). For all practical purposes, we can consider this fixed coordinate frame to be inertial. Note, however, that the ball will dance around wildly if the coordinate frame is instead fixed with respect to the sun. The rotation and revolution of the earth would cause the ball to move at incredible speeds. In reality, inertial frames do not exist; nevertheless, it is a reasonable assumption for earth-based mechanical systems that an inertial frame may be fixed to the earth.

The properties that inertial frames should technically possess are 1) the laws of motions appear the same in any inertial frame, and 2) any frame that moves at constant speed without rotation with respect to an inertial frame is itself inertial. As an example of the second condition, suppose that the racquetball experiment is performed inside of a big truck that is driving along a highway. Ignoring vibrations, if the truck moves at constant speed on a straight stretch of road, then an inertial coordinate frame can be fixed to the truck itself, and the ball will appear to bounce as if the court was not moving. If, however, the road curves or the truck changes its speed, the ball will not bounce the right way. If we still believe that the frame attached to the truck is inertial, then the laws of motion will appear strange. The inertial frame must be attached to the earth in this case to correctly model the behavior of the truck and ball together.

Closed system Another important aspect of the Newton-Euler model is that the system of bodies for which motions are modeled is *closed*, which means that no bodies other than those that are explicitly modeled can have any affect on the motions (imagine, for example, the effect if we forget to account for a black hole

that is a few hundred meters away from the racquetball court).

Newton's laws The motions of bodies are based on three laws that were experimentally verified by Newton and should hold in any inertial frame:

1. An object at rest tends to stay at rest, and an object in motion tends to stay in motion with fixed speed, unless a nonzero resultant[8] force acts upon it.
2. The relationship between a body mass m, its acceleration a, and an applied force f is $f = ma$.
3. The interaction forces between two bodies are of equal magnitude and in opposite directions.

Based on these laws, the differential constraints on a system of moving bodies can be modeled.

13.3.2 Motions of Particles

The Newton-Euler model is described in terms of particles. Each *particle* is considered as a point that has an associated mass m. Forces may act on any particle. The motion of a rigid body, covered in Section 13.3.3, is actually determined by modeling the body as a collection of particles that are stuck together. Therefore, it is helpful to first understand how particles behave.

13.3.2.1 Motion of a single particle

Consider the case of a single particle of mass m that moves in $\mathcal{W} = \mathbb{R}$. The force becomes a scalar, $f \in \mathbb{R}$. Let $q(t)$ denote the position of the particle in $\mathcal{W}$ at time t. Using this notation, acceleration is $\ddot{q}$, and Newton's second law becomes $f = m\ddot{q}$. This can be solved for $\ddot{q}$ to yield

$$\ddot{q} = f/m. \tag{13.50}$$

If f is interpreted as an action variable u, and if $m = 1$, then (13.50) is precisely the double integrator $\ddot{q} = u$ from Example 13.3. Phase variables $x_1 = q$ and $x_2 = \dot{q}$ can be introduced to obtain a state vector $x = (q, \dot{q})$. This means that for a fixed u, the motion of the particle from any initial state can be captured by a vector field on $\mathbb{R}^2$. The state transition equation is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{u}{m}, \end{aligned} \tag{13.51}$$

in which $x_1 = q$, $x_2 = \dot{q}$, and $u = f$. Let $U = [-f_{max}, f_{max}]$, in which f_{max} represents the maximum magnitude of force that can be applied to the particle. Forces

[8] This is the sum of all forces acting on the point.

of arbitrarily high magnitude are not allowed because this would be physically unrealistic.

Now generalize the particle motion to $\mathcal{W} = \mathbb{R}^2$ and $\mathcal{W} = \mathbb{R}^3$. Let n denote the dimension of $\mathcal{W}$, which may be $n = 2$ or $n = 3$. Let q denote the position of the particle in $\mathcal{W}$. Once again, Newton's second law yields $f = m\ddot{q}$, but in this case there are n independent equations of the form $f_i = m\ddot{q}_i$. Each of these may be considered as an independent example of the double integrator, scaled by m. Each component f_i of the force can be considered as an action variable u_i. A $2n$-dimensional state space can be defined as $x = (q, \dot{q})$. The state transition equation for $n = 2$ becomes

$$\begin{aligned} \dot{x}_1 &= x_3 & \qquad \dot{x}_3 &= u_1/m \\ \dot{x}_2 &= x_4 & \qquad \dot{x}_4 &= u_2/m, \end{aligned} \tag{13.52}$$

and for $n = 3$ it becomes

$$\begin{aligned} \dot{x}_1 &= x_4 & \qquad \dot{x}_4 &= u_1/m \\ \dot{x}_2 &= x_5 & \qquad \dot{x}_5 &= u_2/m \\ \dot{x}_3 &= x_6 & \qquad \dot{x}_6 &= u_3/m. \end{aligned} \tag{13.53}$$

For a fixed action, these equations define vector fields on $\mathbb{R}^4$ and $\mathbb{R}^6$, respectively. The action set should also be bounded, as in the one-dimensional case. Suppose that

$$U = \{u \in \mathbb{R}^n \mid \|u\| \leq f_{max}\}. \tag{13.54}$$

Now suppose that multiple forces act on the same particle. In this case, the vector sum

$$F = \sum f \tag{13.55}$$

yields the *resultant force* over all f taken from a collection of forces. The resultant force F represents a single force that is equivalent, in terms of its effect on the particle, to the combined forces in the collection. This enables Newton's second law to be formulated as $F = m\ddot{q}$. The next two examples illustrate state transition equations that arise from a collection of forces, some of which correspond to actions.

Example 13.7 (Lunar Lander) Using the Newton-Euler model of a particle, an example will be constructed for which $X = \mathbb{R}^4$. A lunar lander is modeled as a particle with mass m in a 2D world shown in Figure 13.8. It is not allowed to rotate, implying that $\mathcal{C} = \mathbb{R}^2$. There are three thrusters on the lander, which are on the left, right, and bottom of the lander. The forces acting on the lander are shown in Figure 13.8. The activation of each thruster is considered as a binary switch. Each has its own associated binary action variable, in which the value 1 means that the thruster is firing and 0 means the thruster is dormant. The left and right lateral thrusters provide forces of magnitude f_l and f_r, respectively, when activated (note that the left thruster provides a force to the right, and vice versa).

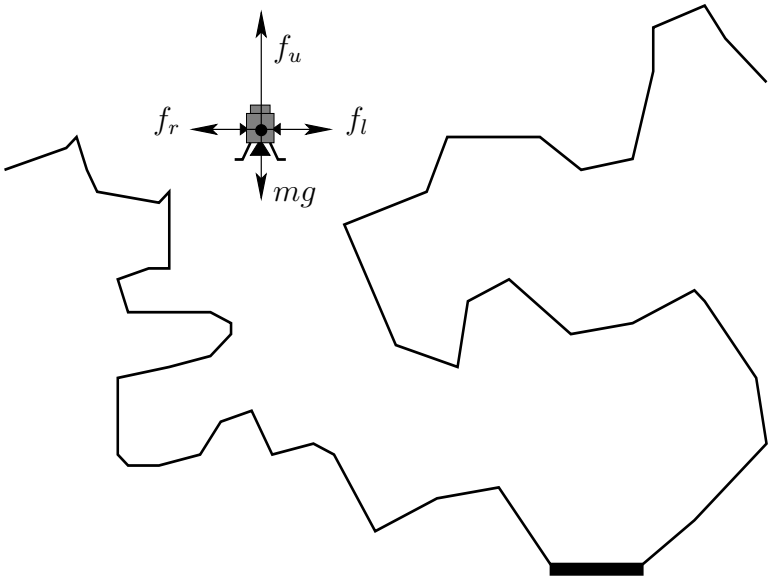


Figure 13.8: There are three thrusters on the lunar lander, and it is under the influence of lunar gravity. It is treated as a particle; therefore, no rotations are possible. Four orthogonal forces may act on the lander: Three arise from thrusters that can be switched on or off, and the remaining arises from the acceleration of gravity.

The upward thruster, mounted to the bottom of the lander, provides a force of magnitude f_u when activated. Let g denote the scalar acceleration constant for gravity (this is approximately 1.622 m/s^2 for the moon).

From (13.55) and Newton's second law, $F = m\ddot{q}$. In the horizontal direction, this becomes

$$m\ddot{q}_1 = u_l f_l - u_r f_r, \tag{13.56}$$

and in the vertical direction,

$$m\ddot{q}_2 = u_u f_u - mg. \tag{13.57}$$

Opposing forces are subtracted because only the magnitudes are given by f_l, f_r, f_u, and g. If they were instead expressed as vectors in $\mathbb{R}^2$, then they would be added.

The lunar lander model can be transformed into a four-dimensional phase space in which $x = (q_1, q_2, \dot{q}_1, \dot{q}_2)$. By replacing $\ddot{q}_1$ and $\ddot{q}_2$ with $\dot{x}_3$ and $\dot{x}_4$, respectively, (13.56) and (13.57) can be written as

$$\dot{x}_3 = \frac{1}{m}(u_l f_l - u_r f_r) \tag{13.58}$$

and

$$\dot{x}_4 = \frac{u_u f_u}{m} - g. \tag{13.59}$$

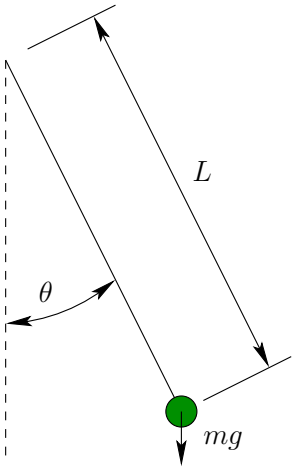


Figure 13.9: The pendulum is a simple and important example of a nonlinear system.

Using $\dot{x}_1 = x_3$ and $\dot{x}_2 = x_4$, the state transition equation becomes

$$\begin{aligned} \dot{x}_1 &= x_3 & \dot{x}_3 &= \frac{f_s}{m}(u_l f_l - u_r f_r) \\ \dot{x}_2 &= x_4 & \dot{x}_4 &= \frac{u_u f_u}{m} - g, \end{aligned} \tag{13.60}$$

which is in the desired form, $\dot{x} = f(x, u)$. The action space U consists of eight elements, which indicate whether each of the three thrusters is turned on or off. Each action vector is of the form (u_l, u_r, u_u), in which each component is 0 or 1. ■

The next example illustrates the importance of Newton's third law.

Example 13.8 (Pendulum) A simple and very important model is the pendulum shown in Figure 13.9. Let m denote the mass of the attached particle (the string is assumed to have no mass). Let g denote the acceleration constant due to gravity. Let L denote the length of the pendulum string. Let θ denote the angular displacement of the pendulum, which characterizes the pendulum configuration. Using Newton's second law and assuming the pendulum moves in a vacuum (no wind resistance), the constraint

$$mL\ddot{\theta} = -mg \sin\theta \tag{13.61}$$

is obtained. A 2D state space can be formulated in which $x_1 = \theta$ and $x_2 = \dot{\theta}$. This leads to

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{L} \sin x_1, \end{aligned} \tag{13.62}$$

which has no actions (the form of (13.62) is $\dot{x} = f(x)$).

A linear drag term $kL\dot{\theta}$ can be added to the model to account for wind resistance. This yields

$$mL\ddot{\theta} = -mg \sin\theta - kL\dot{\theta}, \tag{13.63}$$

which becomes

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{L} \sin x_1 - \frac{k}{m} x_2 \end{aligned} \tag{13.64}$$

in the state space form.

Now consider applying a force u_f on the particle, in a direction perpendicular to the string. This action can be imagined as having a thruster attached to the side of the particle. This adds the term u_f to (13.63). Its sign depends on the choice of the perpendicular vector (thrust to the left or to the right). The state transition equation $\dot{x} = f(x, u)$ then becomes

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{L} \sin x_1 - \frac{k}{m} x_2 + \frac{1}{mL} u_f. \end{aligned} \tag{13.65}$$

■

Although sufficient information has been given to specify differential models for a particle, several other concepts are useful to introduce, especially in the extension to multiple particles and rigid bodies. The main idea is that conservation laws can be derived from Newton's laws. The *linear momentum* (or just *momentum*) d of the particle is defined as

$$d = m\dot{q}. \tag{13.66}$$

This is obtained by integrating $f = m\ddot{q}$ with respect to time.

It will be convenient when rigid-body rotations are covered to work with the *moment of momentum* (or *angular momentum*). A version of momentum that is based on moments can be obtained by first defining the *moment of force* (or *torque*) for a force f acting at a point $q \in W$ as

$$n = q \times f, \tag{13.67}$$

in which $\times$ denotes the vector cross product in $\mathbb{R}^3$. For a particle that has linear momentum d, the *moment of momentum* e is defined as

$$e = q \times d. \tag{13.68}$$

It can be shown that

$$\frac{de}{dt} = n, \tag{13.69}$$

which is equivalent to Newton's second law but is expressed in terms of momentum. For the motion of a particle in a closed system, the linear momentum and moment of momentum are *conserved* if there are no external forces acting on it. This is essentially a restatement of Newton's first law.

This idea can alternatively be expressed in terms of energy, which depends on the same variables as linear momentum. The *kinetic energy* of a particle is

$$T = \frac{1}{2} m\dot{q} \cdot \dot{q}, \tag{13.70}$$

in which $\cdot$ is the familiar inner product (or dot product). The total kinetic energy of a system of particles is obtained by summing the kinetic energies of the individual particles.

13.3.2.2 Motion of a set of particles

The concepts expressed so far naturally extend to a set of particles that move in a closed system. This provides a smooth transition to rigid bodies, which are modeled as a collection of infinitesimal particles that are "stuck together," causing forces between neighboring particles to cancel. In the present model, the particles are independently moving. If a pair of particles collides, then, by Newton's third law, they receive forces of equal magnitude and opposite directions at the instant of impact.

It can be shown that all momentum expressions extend to sums over the particles [681]. For a set of particles, the linear momentum of each can be summed to yield the linear momentum of the system as

$$D = \sum d. \tag{13.71}$$

The total external force can be determined as

$$F = \sum f_i, \tag{13.72}$$

which is a kind of resultant force for the whole system. The relationship $dD/dt = F$ holds, which extends the case of a single particle. The total mass can be summed to yield

$$M = \sum m, \tag{13.73}$$

and the *center of mass* of the system is

$$p = \frac{1}{M} \sum mq, \tag{13.74}$$

in which m and q are the mass and position of each particle, respectively. The expressions $D = M\dot{p}$ and $F = M\ddot{p}$ hold, which are the analogs of $d = m\dot{q}$ and $f = m\ddot{q}$ for a single particle.

So far the translational part of the motion has been captured; however, rotation of the system is also important. This was the motivation for introducing the moment concepts. Let the total moment of force (or total torque) be

$$N = \sum q \times f, \tag{13.75}$$

and let the moment of momentum of the system be

$$E = \sum q \times d. \tag{13.76}$$

It can be shown that $dE/dt = N$, which behaves in the same way as in the single-particle case.

The ideas given so far make a system of particles appear very much as a single particle. It is important, however, when conducting a simulation of their behavior to consider the collisions between the particles. Detecting these collisions and calculating the resulting impact forces ensures that correct motions are obtained.

As the number of particles tends to infinity, consider the limiting case of a rigid body. In this case, the particles are "sewn" together, which cancels their internal forces. It will be sufficient only to handle the forces that act on the boundary of the rigid body. The expressions for the motion of a rigid body are given in Section 13.3.3. The expressions can alternatively be obtained using other concepts, such as those in Section 13.4.

13.3.3 Motion of a Rigid Body

For a free-floating 3D rigid body, recall from Section 4.2.2 that its C-space $\mathcal{C}$ has six dimensions. Suppose that actions are applied to the body as external forces. These directly cause accelerations that result in second-order differential equations. By defining a state to be $(q, \dot{q})$, first-order differential equations can be obtained in a twelve-dimensional phase space X.

Let $\mathcal{A} \subseteq \mathbb{R}^3$ denote a free-floating rigid body. Let $\sigma(r)$ denote the *body density* at $r \in \mathcal{A}$. Let m denote the total mass of $\mathcal{A}$, which is defined using the density as

$$m = \int_{\mathcal{A}} \sigma(r) dr, \tag{13.77}$$

in which $dr = dr_1 dr_2 dr_3$ represents a volume element in $\mathbb{R}^3$. Let $p \in \mathbb{R}^3$ denote the *center of mass* of $\mathcal{A}$, which is defined for $p = (p_1, p_2, p_3)$ as

$$p_i = \frac{1}{m} \int_{\mathcal{A}} r_i \sigma(r) dr. \tag{13.78}$$

Suppose that a collection of external forces acts on $\mathcal{A}$ (it is assumed that all internal forces in $\mathcal{A}$ cancel each other out). Each force f acts at a point on the boundary, as shown in Figure 13.10 (note that any point along the line of force may alternatively be used). The set of forces can be combined into a single force

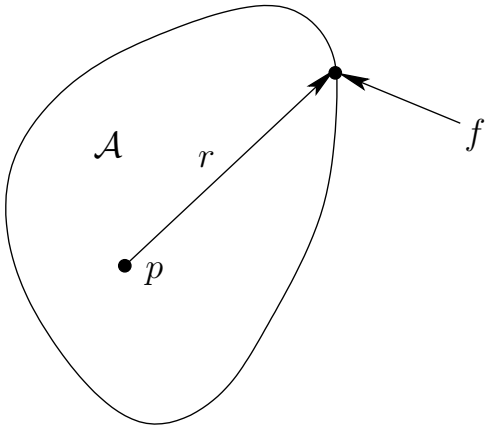


Figure 13.10: A force f acting on $\mathcal{A}$ at r produces a moment about p of $r \times f$.

and moment that both act about the center of mass p. Let F denote the total external force acting on $\mathcal{A}$. Let N denote the total external moment about the center of mass of $\mathcal{A}$. These are given by

$$F = \sum f \tag{13.79}$$

and

$$N = \sum r \times f \tag{13.80}$$

for the collection of external forces. The terms F and N are often called the *resultant force* and *resultant moment* of a collection of forces. It was shown by Poinsot that every system of forces is equivalent to a single force and a moment parallel to the line of action of the force. The result is called a *wrench*, which is the force-based analog of a screw; see [681] for a nice discussion.

Actions of the form $u \in U$ can be expressed as external forces and/or moments that act on the rigid body. For example, a thruster may exert a force on the body when activated. For a given u, the total force and moment can be resolved to obtain $F(u)$ and $N(u)$.

Important frames Three different coordinate frames will become important during the presentation:

1. **Inertial frame:** The global coordinate frame that is fixed with respect to all motions of interest.

2. **Translating frame:** A moving frame that has its origin at the center of mass of $\mathcal{A}$ and its axes aligned with the inertial frame.

3. **Body frame:** A frame that again has its origin at the center of mass of $\mathcal{A}$, but its axes are rigidly attached to $\mathcal{A}$. This is the same frame that was used to define bodies in Chapter 3.

The translational part The state transition equation involves 12 scalar equations. Six of these are straightforward to obtain by characterizing the linear velocity. For this case, it can be imagined that the body does not rotate with respect to the inertial frame. The linear momentum is $D = m\dot{p}$, and Newton's second law implies that

$$F(u) = \frac{dD}{dt} = m\ddot{p}. \tag{13.81}$$

This immediately yields half of the state transition equation by solving for $\ddot{p}$. This yields a 3D version of the double integrator in Example 13.3, scaled by m. Let (p_1, p_2, p_3) denote the coordinates of p. Let (v_1, v_2, v_3) denote the linear velocity the center of mass. Three scalar equations of the state transition equation are $\dot{p}_i = v_i$ for $i = 1, 2, 3$. Three more are obtained as $\dot{v}_i = F_i(u)/m$ for $i = 1, 2, 3$. If there are no moments and the body is not rotating with respect to the inertial frame, then these six equations are sufficient to describe its motion. This may occur for a spacecraft that is initially at rest, and its thrusters apply a total force only through the center of mass.

The rotational part The six equations derived so far are valid even if $\mathcal{A}$ rotates with respect to the inertial frame. They are just the translational part of the motion. The rotational part can be decoupled from the translational part by using the translating frame. All translational aspects of the motion have already been considered. Imagine that $\mathcal{A}$ is only rotating while its center of mass remains fixed. Once the rotational part of the motion has been determined, it can be combined with the translational part by simply viewing things from the inertial frame. Therefore, the motion of $\mathcal{A}$ is now considered with respect to the translating frame, which makes it appear to be pure rotation.

Unfortunately, characterizing the rotational part of the motion is substantially more complicated than the translation case and the 2D rotation case. This should not be surprising in light of the difficulties associated with 3D rotations in Chapters 3 and 4.

Following from Newton's second law, the change in the moment of momentum is

$$N(u) = \frac{dE}{dt}. \tag{13.82}$$

The remaining challenge is to express the right-hand side of (13.82) in a form that can be inserted into the state transition equation.

Differential rotations To express the change in the moment of momentum in detail, the concept of a differential rotation is needed. In the plane, it is straightforward to define $\omega = \dot{\theta}$; however, for $SO(3)$, it is more complicated. One choice is to define derivatives with respect to yaw-pitch-roll variables, but this leads to distortions and singularities, which are problematic for the Newton-Euler formulation. Instead, a differential rotation is defined as shown in Figure 13.11. Let v

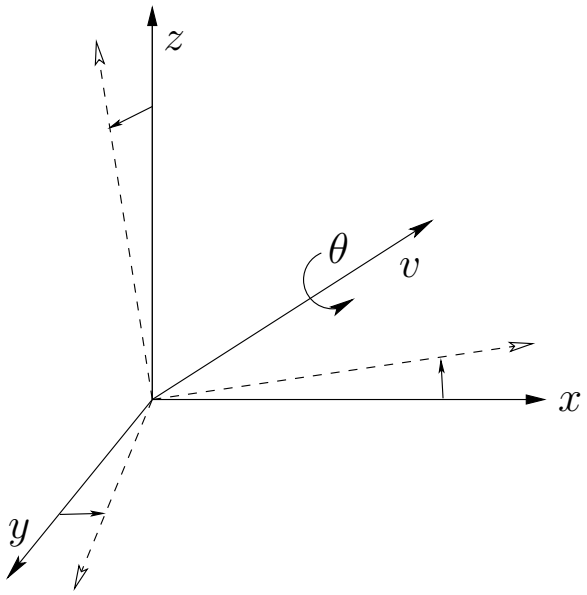


Figure 13.11: The angular velocity is defined as a rotation rate of the coordinate frame about an axis.

denote a unit vector in $\mathbb{R}^3$, and let θ denote a rotation that is analogous to the 2D case. Let ω denote the angular velocity vector,

$$\omega = v\frac{d\theta}{dt}. \tag{13.83}$$

This provides a natural expression for angular velocity.[9] The change in a rotation matrix R with respect to time is

$$\dot{R} = \omega \times R. \tag{13.84}$$

This relationship can be used to derive expressions that relate ω to yaw-pitch-roll angles or quaternions. For example, using the yaw-pitch-roll matrix (3.42) the conversion from ω to the change yaw, pitch, and roll angles is

$$\begin{pmatrix} \dot{\gamma} \\ \dot{\beta} \\ \dot{\alpha} \end{pmatrix} = \frac{1}{\cos\beta} \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha\cos\beta & \cos\alpha\cos\beta & 0 \\ \cos\alpha\sin\beta & \sin\alpha\sin\beta & -\cos\beta \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}. \tag{13.85}$$

Inertia matrix An *inertia matrix* (also called an *inertia tensor* or *inertia operator*) will be derived by considering $\mathcal{A}$ as a collection of particles that are rigidly attached together (all contact forces between them cancel due to Newton's third

[9] One important issue to be aware of is that the integral of ω is not path-invariant (see Example 2.15 of [994]).

law). The expression $\sigma(r)dr$ in (13.77) represents the mass of an infinitesimal particle of $\mathcal{A}$. The *moment of momentum* of the infinitesimal particle is $r \times \dot{r}\sigma(r)dr$. This means that the total moment of momentum of $\mathcal{A}$ is

$$E = \int_{\mathcal{A}(q)} (r \times \dot{r})\, \sigma(r)dr. \tag{13.86}$$

By using the fact that $\dot{r} = \omega \times r$, the expression becomes

$$E = \int_{\mathcal{A}(q)} r \times (\omega \times r)\, \sigma(r)dr. \tag{13.87}$$

Observe that r now appears twice in the integrand. By doing some algebraic manipulations, ω can be removed from the integrand, and a function that is quadratic in the r variables is obtained (since r is a vector, the function is technically a quadratic form). The first step is to apply the identity $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$ to obtain

$$E = \int_{\mathcal{A}(q)} \big((r \cdot r)\omega - (r \cdot \omega)r\big)\sigma(r)dr. \tag{13.88}$$

The angular velocity can be moved to the right to obtain

$$E = \left(\int_{\mathcal{A}(q)} \big((r \cdot r)I_3 - rr^T\big)\sigma(r)dr \right) \omega, \tag{13.89}$$

in which the integral now occurs over a 3×3 matrix and I_3 is the 3×3 identity matrix.

Let I be called the *inertia matrix* and be defined as

$$I(q) = \left(\int_{\mathcal{A}(q)} \big((r \cdot r)I_3 - rr^T\big)\sigma(r)dr \right). \tag{13.90}$$

Using the definition,

$$E = I\omega. \tag{13.91}$$

This simplification enables a concise expression of (13.82) as

$$N(u) = \frac{dE}{dt} = \frac{d(I\omega)}{dt} = I\frac{d\omega}{dt} + \frac{dI}{dt}\omega, \tag{13.92}$$

which makes use of the chain rule.

Simplifying the inertia matrix Now the inertia matrix will be considered more carefully. It is a symmetric 3×3 matrix, which can be expressed as

$$I(q) = \begin{pmatrix} I_{11}(q) & I_{12}(q) & I_{13}(q) \\ I_{12}(q) & I_{22}(q) & I_{23}(q) \\ I_{13}(q) & I_{23}(q) & I_{33}(q) \end{pmatrix}. \tag{13.93}$$

For each $i \in \{1, 2, 3\}$, the entry $I_{ii}(q)$ is called a *moment of inertia*. The three cases are

$$I_{11}(q) = \int_{\mathcal{A}(q)} (r_2^2 + r_3^2)\sigma(r)dr, \tag{13.94}$$

$$I_{22}(q) = \int_{\mathcal{A}(q)} (r_1^2 + r_3^2)\sigma(r)dr, \tag{13.95}$$

and

$$I_{33}(q) = \int_{\mathcal{A}(q)} (r_1^2 + r_2^2)\sigma(r)dr. \tag{13.96}$$

The remaining entries are defined as follows. For each $i, j \in \{1, 2, 3\}$ such that $i \neq j$, the *product of inertia* is

$$H_{ij}(q) = \int_{\mathcal{A}(q)} r_i r_j \sigma(r)dr, \tag{13.97}$$

and $I_{ij}(q) = -H_{ij}(q)$.

One problem with the formulation so far is that the inertia matrix changes as the body rotates because all entries depend on the orientation q. Recall that it was derived by considering $\mathcal{A}$ as a collection of infinitesimal particles in the translating frame. It is possible, however, to express the inertia matrix in the body frame of $\mathcal{A}$. In this case, the inertia matrix can be denoted as I because it does not depend on the orientation of $\mathcal{A}$ with respect to the translational frame. The original inertia matrix is then recovered by applying a rotation that relates the body frame to the translational frame: $I(q) = RI$, in which R is a rotation matrix. It can be shown (see Equation (2.91) and Section 3.2 of [994]) that after performing this substitution, (13.92) simplifies to

$$N(u) = I\frac{d\omega}{dt} + \omega \times (I\omega). \tag{13.98}$$

The body frame of $\mathcal{A}$ must have its origin at the center of mass p; however, its orientation has not been constrained. For different orientations, different inertia matrices will be obtained. Since I captures the physical characteristics of $\mathcal{A}$, any two inertia matrices differ only by a rotation. This means for a given $\mathcal{A}$, all inertia matrices that can be defined by different body frame orientations have the same eigenvalues and eigenvectors. Consider the positive definite quadratic form $x^T Ix = 1$, which represents the equation of an ellipsoid. A standard technique in linear algebra is to compute the principle axes of an ellipsoid, which turn out to be the eigenvectors of I. The lengths of the ellipsoid axes are given by the eigenvalues. An axis-aligned expression of the ellipsoid can be obtained by defining $x' = Rx$, in which R is the matrix formed by columns of eigenvectors. Therefore, there exists an orientation of the body frame in which the inertia matrix simplifies to

$$I = \begin{pmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix} \tag{13.99}$$

and the diagonal elements are the eigenvalues. If the body happens to be an ellipsoid, the principle axes correspond to the ellipsoid axes. Moment of inertia tables are given in many texts [690]; in these cases, the principle axes are usually chosen as the axis of the body frame because they result in the simplest expression of I.

Completing the state transition equation Assume that the body frame of $\mathcal{A}$ aligns with the principle axes. The remaining six equations of motion can finally be given in a nice form. Using (13.99), the expression (13.98) reduces to [681]

$$\begin{pmatrix} N_1(u) \\ N_2(u) \\ N_3(u) \end{pmatrix} = \begin{pmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix} \begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix} + \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \begin{pmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}. \tag{13.100}$$

Multiplying out (13.100) yields

$$\begin{aligned} N_1(u) &= I_{11}\dot{\omega}_1 + (I_{33} - I_{22})\omega_2\omega_3 \\ N_2(u) &= I_{22}\dot{\omega}_2 + (I_{11} - I_{33})\omega_3\omega_1 \\ N_3(u) &= I_{33}\dot{\omega}_3 + (I_{22} - I_{11})\omega_1\omega_2. \end{aligned} \tag{13.101}$$

To prepare for the state transition equation form, solving for $\dot{\omega}$ yields

$$\begin{aligned} \dot{\omega}_1 &= \big(N_1(u) + (I_{22} - I_{33})\omega_2\omega_3\big)/I_{11} \\ \dot{\omega}_2 &= \big(N_2(u) + (I_{33} - I_{11})\omega_3\omega_1\big)/I_{22} \\ \dot{\omega}_3 &= \big(N_3(u) + (I_{11} - I_{22})\omega_1\omega_2\big)/I_{33}. \end{aligned} \tag{13.102}$$

One final complication is that ω needs to be related to angles that are used to express an element of $SO(3)$. The mapping between these depends on the particular parameterization of $SO(3)$. Suppose that quaternions of the form (a, b, c, d) are used to express rotation. Recall that a can be recovered once b, c, and d are given using $a^2 + b^2 + c^2 + d^2 = 1$. The relationship between ω and the time derivatives of the quaternion components is obtained by using (13.84) (see [690], p. 433):

$$\begin{aligned} \dot{b} &= \omega_3 c - \omega_2 d \\ \dot{c} &= \omega_1 d - \omega_3 b \\ \dot{d} &= \omega_2 b - \omega_1 c. \end{aligned} \tag{13.103}$$

This finally completes the specification of $\dot{x} = f(x, u)$, in which

$$x = (p_1, p_2, p_3, v_1, v_2, v_3, b, c, d, \omega_1, \omega_2, \omega_3) \tag{13.104}$$

is a twelve-dimensional phase vector. For convenience, the full specification of the state transition equation is

$$\begin{aligned}
\dot{p}_1 &= v_1 & \dot{b} &= \omega_3 c - \omega_2 d \\
\dot{p}_2 &= v_2 & \dot{c} &= \omega_1 d - \omega_3 b \\
\dot{p}_3 &= v_3 & \dot{d} &= \omega_2 b - \omega_1 c \\
\dot{v}_1 &= F_1(u)/m & \dot{\omega}_1 &= \big(N_1(u) + (I_{22} - I_{33})\omega_2\omega_3\big)/I_{11} \\
\dot{v}_2 &= F_2(u)/m & \dot{\omega}_2 &= \big(N_2(u) + (I_{33} - I_{11})\omega_3\omega_1\big)/I_{22} \\
\dot{v}_3 &= F_3(u)/m & \dot{\omega}_3 &= \big(N_3(u) + (I_{11} - I_{22})\omega_1\omega_2\big)/I_{33}.
\end{aligned} \tag{13.105}$$

The relationship between inertia matrices and ellipsoids is actually much deeper than presented here. The kinetic energy due to rotation only is elegantly expressed as

$$T = \tfrac{1}{2}\omega^T I \omega. \tag{13.106}$$

A fascinating interpretation of rotational motion in the absence of external forces was given by Poinsot [39, 681]. As the body rotates, its motion is equivalent to that of the inertia ellipsoid, given by (13.106), rolling (without sliding) down a plane with normal vector $I\omega$ in $\mathbb{R}^3$.

The 2D case The dynamics of a 2D rigid body that moves in the plane can be handled as a special case of a 3D body. Let $\mathcal{A} \subset \mathbb{R}^2$ be a 2D body, expressed in its body frame. The total external forces acting on $\mathcal{A}$ can be expressed in terms of a two-dimensional total force through the center of mass and a moment through the center of mass. The phase space for this model has six dimensions. Three come from the degrees of freedom of $SE(2)$, two come from linear velocity, and one comes from angular velocity.

The translational part is once again expressed as

$$F(u) = \frac{dD}{dt} = m\ddot{p}. \tag{13.107}$$

This provides four components of the state transition equation.

All rotations must occur with respect to the z-axis in the 2D formulation. This means that the angular velocity ω is a scalar value. Let θ denote the orientation of $\mathcal{A}$. The relationship between ω and θ is given by $\dot{\theta} = \omega$, which yields one more component of the state transition equation.

At this point, only one component remains. Recall (13.92). By inspecting (13.101) it can be seen that the inertia-based terms vanish. In that formulation, ω_3 is equivalent to the scalar ω for the 2D case. The final terms of all three equations vanish because $\omega_1 = \omega_2 = 0$. The first terms of the first two equations also vanish because $\dot{\omega}_1 = \dot{\omega}_2 = 0$. This leaves $N_3(u) = I_{33}\dot{\omega}_3$. In the 2D case, this can be notationally simplified to

$$N(u) = \frac{dE}{dt} = \frac{d(I\omega)}{dt} = I\frac{d\omega}{dt} = I\dot{\omega}, \tag{13.108}$$

in which I is now a scalar. Note that for the 3D case, the angular velocity can change, even when $N(u) = 0$. In the 2D case, however, this is not possible. In both cases, the moment of momentum is conserved; in the 2D case, this happens to imply that ω is fixed. The sixth component of the state transition equation is obtained by solving (13.108) for $\dot{\omega}$.

The state transition equation for a 2D rigid body in the plane is therefore

$$\begin{aligned} \dot{p}_1 &= v_1 & \dot{v}_1 &= F_1(u)/m \\ \dot{p}_2 &= v_2 & \dot{v}_2 &= F_2(u)/m \\ \dot{\theta} &= \omega & \dot{\omega} &= N(u)/I. \end{aligned} \tag{13.109}$$

A car with tire skidding This section concludes by introducing a car model that considers it as a skidding rigid body in the plane. This model was suggested by Jim Bernard. The C-space is $\mathcal{C} = \mathbb{R}^2 \times \mathbb{S}^1$, in which $q = (x, y, \theta)$. Suppose that as the car moves at high speeds, the tires are able to skid laterally in a direction perpendicular to the main axis of the car (i.e., parallel to the rear axle). Let ω denote the angular velocity of the car. Let v denote the lateral skidding velocity, which is another state variable. This results in a five-dimensional state space in which each state is a vector of the form $(x, y, \theta, \omega, v)$.

The position of the rear axle center can be expressed as

$$\begin{aligned} \dot{x} &= s\cos\theta - v\sin\theta \\ \dot{y} &= s\sin\theta + v\cos\theta, \end{aligned} \tag{13.110}$$

which yields two components of the state transition equation. Let $\omega = \dot{\theta}$ denote the angular velocity, which yields one more component of the state transition equation. This leaves only two equations, which are derived from 2D rigid body mechanics (which will be covered in Section 13.3.3). The state transition is

$$\begin{aligned} \dot{x} &= s\cos\theta - v\sin\theta \\ \dot{y} &= s\sin\theta + v\cos\theta \\ \dot{\theta} &= \omega \\ \dot{\omega} &= (af_f - bf_r)/I \\ \dot{v} &= -s\omega + (f_f + f_r)/m, \end{aligned} \tag{13.111}$$

in which f_f and f_r are the front and rear tire forces, m is the mass, I is the moment of inertia, and a and b are the distances from the center of mass to the front and rear axles, respectively. The first force is

$$f_f = c_f\big((v + a\omega)/s + \phi\big), \tag{13.112}$$

in which c_f is the front cornering stiffness, and ϕ is the steering angle. The second force is

$$f_r = c_r(v - b\omega)/s, \tag{13.113}$$

in which c_r is the rear cornering stiffness. The steering angle can be designated as an action variable: $u_\phi = \phi$. An integrator can be placed in front of the speed to allow accelerations. This increases the state space dimension by one.

Reasonable values for the parameters for an automotive application are: $m =$ 1460 kg, $c_f = 17000$, $c_r = 20000$, $a = 1.2$ m, $b = 1.5$ m, $I = 2170$ kg/m^2, and $s = 27$ m/sec. This state transition equation involves a linear tire skidding model, which is a poor approximation in many applications. Nonlinear tire models provide better approximations to the actual behavior of cars [91]. For a thorough introduction to the dynamics of cars, see [822].

13.4 Advanced Mechanics Concepts

Newton-Euler mechanics has the advantage that it starts with very basic principles, but it has frustrating restrictions that make modeling more difficult for complicated mechanical systems. One of the main limitations is that all laws must be expressed in terms of an inertial frame with orthogonal axes. This section introduces the basic ideas of Lagrangian and Hamiltonian mechanics, which remove these restrictions by reducing mechanics to finding an optimal path using any coordinate neighborhood of the C-space. The optimality criterion is expressed in terms of energy. The resulting techniques can be applied on any coordinate neighborhood of a smooth manifold. The Lagrangian formulation is usually best for determining the motions of one or more bodies. Section 13.4.1 introduces the basic Lagrangian concepts based on the calculus of variations. Section 13.4.2 presents a general form of the Euler-Lagrange equations, which is useful for determining the motions of numerous dynamical systems, including chains of bodies. The Lagrangian is also convenient for systems that involve additional differential constraints, such as friction or rolling wheels. These cases are briefly covered in Section 13.4.3. The Hamiltonian formulation in Section 13.4.4 is based on a special phase space and provides an alternative to the Lagrangian formulation. The technique generalizes to Pontryagin's minimum principle, a powerful optimal control technique that is covered in Section 15.2.3.

13.4.1 Lagrangian Mechanics

13.4.1.1 Calculus of variations

Lagrangian mechanics is based on the *calculus of variations*, which is the subject of optimization over a space of paths. One of the most famous variational problems involves constraining a particle to travel along a curve (imagine that the particle slides along a frictionless track). The problem is to find the curve for which the ball travels from one point to the other, starting at rest, and being accelerated only by gravity. The solution is a cycloid function called the *Brachistochrone curve* [841]. Before this problem is described further, recall the classical optimization problem from calculus in which the task is to find extremal values (minima and

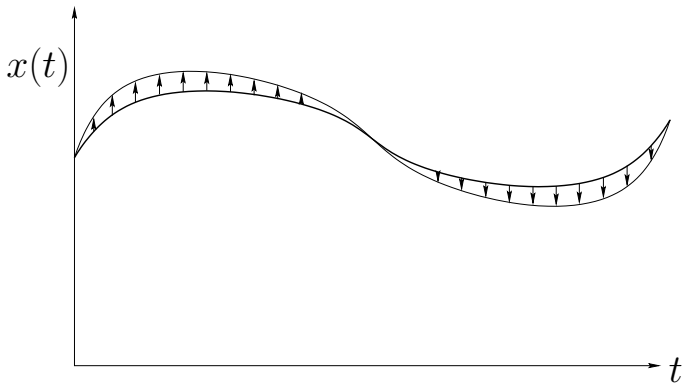


Figure 13.12: The variation is a "small" function that is added to $\tilde{x}$ to perturb it.

maxima) of a function. Let $\tilde{x}$ denote a smooth function from $\mathbb{R}$ to $\mathbb{R}$, and let $x(t)$ denote its value for any $t \in \mathbb{R}$. From standard calculus, the extremal values of $\tilde{x}$ are all $t \in \mathbb{R}$ for which $\dot{x} = 0$. Suppose that at some $t' \in \mathbb{R}$, $\tilde{x}$ achieves a local minimum. To serve as a local minimum, tiny perturbations of t' should result in larger function values. Thus, there exists some $d > 0$ such that $x(t' + \epsilon) > x(t')$ for any $\epsilon \in [-d, d]$. Each ϵ represents a possible perturbation of t'.

The calculus of variations addresses a harder problem in which optimization occurs over a space of functions. For each function, a value is assigned by a criterion called a *functional*.[10] A procedure analogous to taking the derivative of the function and setting it to zero will be performed. This will be arrived at by considering tiny perturbations of an entire function, as opposed to the ϵ perturbations mentioned above. Each perturbation is itself a function, which is called a *variation*. For a function to minimize a functional, any small enough perturbation of it must yield a larger functional value. In the case of optimizing a function of one variable, there are only two directions for the perturbation: $\pm\epsilon$. See Figure 13.12. In the calculus of variations, there are many different "directions" because of the uncountably infinite number of ways to construct a small variation function that perturbs the original function (the set of all variations is an infinite-dimensional function space; recall Example 8.5).

Let $\tilde{x}$ denote a smooth function from $T = [t_0, t_1]$ into $\mathbb{R}$. The functional is defined by integrating a function over the domain of $\tilde{x}$. Let L be a smooth, real-valued function of three variables, a, b, and c.[11] The arguments of L may be any $a, b \in \mathbb{R}$ and $c \in T$ to yield $L(a, b, c)$, but each has a special interpretation. For some smooth function $\tilde{x}$, L is used to evaluate it at a particular $t \in T$ to obtain $L(x, \dot{x}, t)$. A *functional* Φ is constructed using L to evaluate the whole function $\tilde{x}$

[10]This is the reason why a cost *functional* has been used throughout the book. It is a function on a space of functions.

[11]Unfortunately, L is used here to represent a cost function, on which a functional Φ will be based. This conflicts with using l as a cost function and L as the functional in motion planning formulations. This notational collision remains because L is standard notation for the Lagrangian. Be careful to avoid confusion.

as

$$\Phi(\tilde{x}) = \int_T L(x(t), \dot{x}(t), t) dt. \tag{13.114}$$

The problem is to select an $\tilde{x}$ that optimizes Φ. The approach is to take the derivative of Φ and set it equal to zero, just as in standard calculus; however, differentiating Φ with respect to $\tilde{x}$ is not standard calculus. This usually requires special conditions on the class of possible functions (e.g., smoothness) and on the vector space of variations, which are implicitly assumed to hold for the problems considered in this section.

Example 13.9 (Shortest-Path Functional) As an example of a functional, consider

$$L(x, \dot{x}, t) = \sqrt{1 + \dot{x}^2}. \tag{13.115}$$

When evaluated on a function $\tilde{x}$, this yields the arc length of the path. ■

Another example of a functional has already been seen in the context of motion planning. The cost functional (8.39) assigns a cost to a path taken through the state space. This provided a natural way to formulate optimal path planning. A discrete, approximate version was given by (7.26).

Let h be a smooth function over T, and let $\epsilon \in \mathbb{R}$ be a small constant. Consider the function defined as $x(t) + \epsilon h(t)$ for all $t \in [0, 1]$. If $\epsilon = 0$, then (13.114) remains the same. As ϵ is increased or decreased, then $\Phi(\tilde{x} + \epsilon h)$ may change. The function h is like the "direction" in a directional derivative. If for any smooth function h, their exists some $\epsilon > 0$ such that the value $\Phi(\tilde{x} + \epsilon h)$ increases, then $\tilde{x}$ is called an *extremal* of Φ. Any small perturbation to $\tilde{x}$ causes the value of Φ to increase. Therefore, $\tilde{x}$ behaves like a local minimum in a standard optimization problem.

Let $g = \epsilon h$ for some $\epsilon > 0$ and function h. The differential of a functional can be approximated as [39]

$$\begin{aligned}
\Phi(\tilde{x} + g) - \Phi(\tilde{x}) &= \int_T \Big(L(x(t) + g(t), \dot{x}(t) + \dot{g}(t), t) - L(x(t), \dot{x}(t), t) \Big) dt + \cdots \\
&= \int_T \left(\frac{\partial L}{\partial x} g + \frac{\partial L}{\partial \dot{x}} \dot{g} \right) dt + \cdots \\
&= \int_T \left(\frac{\partial L}{\partial x} g - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} g \right) dt + \left(\frac{\partial L}{\partial \dot{x}} g \right) \Bigg|_{t_0}^{t_1} + \cdots ,
\end{aligned} \tag{13.116}$$

in which $\cdots$ represents higher order terms that will vanish in the limit. The last step follows from integration by parts:

$$\left(\frac{\partial L}{\partial \dot{x}} g \right) \Bigg|_{t_0}^{t_1} = \int_T \frac{\partial L}{\partial \dot{x}} \dot{g} dt + \int_T \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} h dt, \tag{13.117}$$

which is just $uv = \int v du + \int u dv$. Consider the value of (13.116) as ϵ becomes small, and assume that $h(t_0) = h(t_1) = 0$. For $\tilde{x}$ to be an extremal function, the change expressed in (13.116) should tend to zero as the variations approach zero. Based on further technical assumptions, including the *Fundamental Lemma of the Calculus of Variations* (see Section 12 of [39]), the *Euler-Lagrange equation*,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0, \tag{13.118}$$

is obtained as a necessary condition for $\tilde{x}$ to be an extremum. Intuition can be gained by studying the last line of (13.116). The integral attains a zero value precisely when (13.118) is satisfied. The other terms vanish because $h(t_0) = h(t_1) = 0$, and higher order terms disappear in the limit process.

The partial derivatives of L with respect to $\dot{x}$ and x are defined using standard calculus. The derivative $\partial L/\partial \dot{x}$ is evaluated by treating $\dot{x}$ as an ordinary variable (i.e., as $\partial L/\partial b$ when the variables are named as in $L(a, b, c)$). Following this, the derivative of $\partial L/\partial \dot{x}$ with respect to t is taken. To illustrate this process, consider the following example.

Example 13.10 (A Simple Variational Problem) Let L be a functional defined as

$$L(x, \dot{x}, t) = x^3 + \dot{x}^2. \tag{13.119}$$

The partial derivatives with respect to x and $\dot{x}$ are

$$\frac{\partial L}{\partial x} = 3x^2 \tag{13.120}$$

and

$$\frac{\partial L}{\partial \dot{x}} = 2\dot{x}. \tag{13.121}$$

Taking the time derivative of (13.121) yields

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 2\ddot{x} \tag{13.122}$$

Substituting these into the Euler-Lagrange equation (13.118) yields

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 2\ddot{x} - 3x^2 = 0. \tag{13.123}$$

This represents a second-order differential constraint that constrains the acceleration as $\ddot{x} = 3x^2/2$. By constructing a 2D phase space, the constraint could be expressed using first-order differential equations. ■

13.4.1.2 Hamilton's principle of least action

Now sufficient background has been given to return to the dynamics of mechanical systems. The path through the C-space of a system of bodies can be expressed as the solution to a calculus of variations problem that optimizes the difference between kinetic and potential energy. The calculus of variations principles generalize to any coordinate neighborhood of $\mathcal{C}$. In this case, the Euler-Lagrange equation is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0, \tag{13.124}$$

in which q is a vector of n coordinates. It is actually n scalar equations of the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0. \tag{13.125}$$

The coming presentation will use (13.124) to obtain a phase transition equation. This will be derived by optimizing a functional defined as the change in kinetic and potential energy. Kinetic energy for particles and rigid bodies was defined in Section 13.3.1. In general, the kinetic energy function must be a quadratic function of $\dot{q}$. Its definition can be interpreted as an inner product on $\mathcal{C}$, which causes $\mathcal{C}$ to become a *Riemannian manifold* [156]. This gives the manifold a notion of the "angle" between velocity vectors and leads to well-defined notions of curvature and shortest paths called *geodesics*. Let $K(q, \dot{q})$ denote the kinetic energy, expressed using the manifold coordinates, which always takes the form

$$K(q, \dot{q}) = \tfrac{1}{2}\dot{q}^T M(q)\dot{q}, \tag{13.126}$$

in which $M(q)$ is an $n \times n$ matrix called the *mass matrix* or *inertia matrix*.

The next step is to define potential energy. A system is called *conservative* if the forces acting on a point depend only on the point's location, and the work done by the force along a path depends only on the endpoints of the path. The total energy is conserved under the motion of a conservative system. In this case, there exists a *potential function* $\phi : W \to \mathbb{R}$ such that $F = \partial\phi/\partial p$, for any $p \in W$. Let $V(q)$ denote the total *potential energy* of a collection of bodies, placed at configuration q.

It will be assumed that the dynamics are time-invariant. *Hamilton's principle of least action* states that the trajectory, $\tilde{q} : T \to \mathcal{C}$, of a mechanical system coincides with extremals of the functional,

$$\Phi(\tilde{q}) = \int_T \Big(K(q(t), \dot{q}(t)) - V(q(t)) \Big) dt, \tag{13.127}$$

using *any* coordinate neighborhood of $\mathcal{C}$. The principle can be seen for the case of $\mathcal{C} = \mathbb{R}^3$ by expressing Newton's second law in a way that looks like (13.124) [39]:

$$\frac{d}{dt}(m\dot{q}) - \frac{\partial V}{\partial q} = 0, \tag{13.128}$$

in which the force is replaced by the derivative of potential energy. This suggests applying the Euler-Lagrange equation to the functional

$$L(q, \dot{q}) = K(q, \dot{q}) - V(q), \tag{13.129}$$

in which it has been assumed that the dynamics are time-invariant; hence, $L(q, \dot{q}, t) = L(q, \dot{q})$. Applying the Euler-Lagrange equation to (13.127) yields the extremals.

The advantage of the Lagrangian formulation is that the C-space does not have to be $\mathcal{C} = \mathbb{R}^3$, described in an inertial frame. The Euler-Lagrange equation gives a necessary condition for the motions in any C-space of a mechanical system. The conditions can be expressed in terms of any coordinate neighborhood, as opposed to orthogonal coordinate systems, which are required by the Newton-Euler formulation. In mechanics literature, the q variables are often referred to as *generalized coordinates*. This simply means the coordinates given by any coordinate neighborhood of a smooth manifold.

Thus, the special form of (13.124) that uses (13.129) yields the appropriate constraints on the motion:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial K(q, \dot{q})}{\partial \dot{q}} - \frac{\partial K(q, \dot{q})}{\partial q} + \frac{\partial V(q)}{\partial q} = 0. \tag{13.130}$$

Recall that this represents n equations, one for each coordinate q_i. Since $K(q, \dot{q})$ does not depend on time, the d/dt operator simply replaces $\dot{q}$ by $\ddot{q}$ in the calculated expression for $\partial K(q, \dot{q})/\partial \dot{q}$. The appearance of $\ddot{q}$ seems appropriate because the resulting differential equations are second-order, which is consistent with Newton-Euler mechanics.

Example 13.11 (A Falling Particle) Suppose that a particle with mass m is falling in $\mathbb{R}^3$. Let (q_1, q_2, q_3) denote the position of the particle. Let g denote the acceleration constant of gravity in the $-q_3$ direction. The potential energy is $V(q) = mgq_3$. The kinetic energy is

$$K(q, \dot{q}) = \tfrac{1}{2} m \dot{q} \cdot \dot{q} = \tfrac{1}{2} m (\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2). \tag{13.131}$$

The Lagrangian is

$$L(q, \dot{q}) = K(q, \dot{q}) - V(q) = \tfrac{1}{2} m (\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - mgq_3 = 0. \tag{13.132}$$

To obtain the differential constraints on the motion of the particle, use (13.130). For each i from 1 to 3,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{d}{dt}(m\dot{q}_i) = m\ddot{q}_i \tag{13.133}$$

Since $K(q, \dot{q})$ does not depend on q, the derivative $\partial K/\partial q_i = 0$ for each i. The derivatives with respect to potential energy are

$$\frac{\partial V}{\partial q_1} = 0 \qquad \frac{\partial V}{\partial q_2} = 0 \qquad \frac{\partial V}{\partial q_3} = mg. \tag{13.134}$$

Substitution into (13.130) yields three equations:

$$m\ddot{q}_1 = 0 \qquad m\ddot{q}_2 = 0 \qquad m\ddot{q}_3 + mg = 0. \tag{13.135}$$

These indicate that acceleration only occurs in the $-q_3$ direction, and this is due to gravity. The equations are consistent with Newton's laws. As usual, a six-dimensional phase space can be defined to obtain first-order differential constraints. ■

The "least" part of Hamilton's principle is actually a misnomer. It is technically only a principle of "extremal" action because (13.130) can also yield motions that maximize the functional.

13.4.1.3 Applying actions

Up to this point, it has been assumed that no actions are applied to the mechanical system. This is the way the Euler-Lagrange equation usually appears in physics because the goal is to predict motion, rather than control it. Let $u \in \mathbb{R}^n$ denote an action vector. Actions can be applied to the Lagrangian formulation as *generalized forces* that "act" on the right side of the Euler-Lagrange equation. This results in

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = u. \tag{13.136}$$

The actions force the mechanical system to deviate from its usual behavior. In some instances, the true actions may be expressed in terms of other variables, and then u is obtained by a transformation (recall transforming action variables for the differential drive vehicle of Section 13.1.2). In this case, u may be replaced in (13.136) by $\phi(u)$ for some transformation ϕ. In this case, the dimension of u need not be n.

13.4.1.4 Procedure for deriving the state transition equation

The following general procedure can be followed to derive the differential model using Lagrangian mechanics on a coordinate neighborhood of a smooth n-dimensional manifold:

1. Determine the degrees of freedom of the system and define the appropriate n-dimensional smooth manifold $\mathcal{C}$.

2. Express the kinetic energy as a quadratic form in the configuration velocity components:

$$K(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}(q)\dot{q}_i\dot{q}_j. \tag{13.137}$$

3. Express the potential energy $V(q)$.

4. Let $L(q, \dot{q}) = K(q, \dot{q}) - V(q)$ be the Lagrangian function, and use the Euler-Lagrange equation (13.130) to determine the differential constraints.

5. Convert to phase space form by letting $x = (q, \dot{q})$. If possible, solve for $\dot{x}$ to obtain $\dot{x} = f(x, u)$.

Example 13.12 (2D Rigid Body Revisited) The equations in (13.109) can be alternatively derived using the Euler-Lagrange equation. Let $\mathcal{C} = \mathbb{R}^2 \times \mathbb{S}^1$, and let $(q_1, q_2, q_3) = (x, y, \theta)$ to conform to the notation used to express the Lagrangian.

The kinetic energy is the sum of kinetic energies due to linear and angular velocities, respectively. This yields

$$K(q, \dot{q}) = \tfrac{1}{2} m \dot{q} \cdot \dot{q} + \tfrac{1}{2} I \dot{q}_3^2, \tag{13.138}$$

in which m and I are the mass and moment of inertia, respectively. Assume there is no gravity; hence, $V(q) = 0$ and $L(q, \dot{q}) = K(q, \dot{q})$.

Suppose that generalized forces u_1, u_2, and u_3 can be applied to the configuration variables. Applying the Euler-Lagrange equation to $L(q, \dot{q})$ yields

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} &= \frac{d}{dt}(m\dot{q}_1) = m\ddot{q}_1 = u_1 \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} &= \frac{d}{dt}(m\dot{q}_2) = m\ddot{q}_2 = u_2 \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_3} - \frac{\partial L}{\partial q_3} &= \frac{d}{dt}(I\dot{q}_3) = I\ddot{q}_3 = u_3.
\end{aligned} \tag{13.139}$$

These expressions are equivalent to those given in (13.109). One difference is that conversion to the phase space is needed. The second difference is that the action variables in (13.139) do not refer directly to forces and moments. They are instead interpreted as generalized forces that act on the configuration variables. A conversion should be performed if the original actions in (13.109) are required. ■

13.4.2 General Lagrangian Expressions

As more complicated mechanics problems are considered, it is convenient to express the differential constraints in a general form. For example, evaluating (13.130) for a kinematic chain of bodies leads to very complicated expressions. The terms of these expressions, however, can be organized into standard forms that appear simpler and give some intuitive meanings to the components.

Suppose that the kinetic energy is expressed using (13.126), and let $m_{ij}(q)$ denote an entry of $M(q)$. Suppose that the potential energy is $V(q)$. By performing the derivatives expressed in (13.136), the Euler-Lagrange equation can be

expressed as n scalar equations of the form [856]

$$\sum_{j=1}^{n} m_{ij}(q)\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n} h_{ijk}(q)\dot{q}_j\dot{q}_k + g_i(q) = u_i \tag{13.140}$$

in which

$$h_{ijk} = \frac{\partial m_{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial m_{jk}}{\partial q_i}. \tag{13.141}$$

There is one equation for each i from 1 to n. The components of (13.140) have physical interpretations. The m_{ii} coefficients represent the inertia with respect to q_i. The m_{ij} represent the affect on q_j of accelerating q_i. The $h_{ijj}\dot{q}_j^2$ terms represent the centrifugal effect induced on q_i by the velocity of q_j. The $h_{ijk}\dot{q}_j\dot{q}_k$ terms represent the Coriolis effect induced on q_i by the velocities of q_j and q_k. The g_i term usually arises from gravity.

An alternative to (13.140) is often given in terms of matrices. It can be shown that the Euler-Lagrange equation reduces to

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u, \tag{13.142}$$

which represents n scalar equations. This introduces $C(q,\dot{q})$, which is an $n \times n$ *Coriolis matrix*. It turns out that many possible Coriolis matrices may produce equivalent different constraints. With respect to (13.140), the Coriolis matrix must be chosen so that

$$\sum_{j=1}^{n} c_{ij}\dot{q}_j = \sum_{j=1}^{n}\sum_{k=1}^{n} h_{ijk}\dot{q}_j\dot{q}_k. \tag{13.143}$$

Using (13.141),

$$\sum_{j=1}^{n} c_{ij}\dot{q}_j = \sum_{j=1}^{n}\sum_{k=1}^{n} \left(\frac{\partial m_{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial m_{jk}}{\partial q_i} \right) \dot{q}_j\dot{q}_k. \tag{13.144}$$

A standard way to determine $C(q,\dot{q})$ is by computing *Christoffel symbols*. By subtracting $\frac{1}{2}\frac{\partial m_{jk}}{\partial q_i}$ from the inside of the nested sums in (13.144), the equation can be rewritten as

$$\sum_{j=1}^{n} c_{ij}\dot{q}_j = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial m_{ij}}{\partial q_k}\dot{q}_j\dot{q}_k + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \left(\frac{\partial m_{ij}}{\partial q_k} - \frac{\partial m_{jk}}{\partial q_i} \right) \dot{q}_j\dot{q}_k. \tag{13.145}$$

This enables an element of $C(q,\dot{q})$ to be written as

$$c_{ij} = \sum_{k=1}^{n} c_{ijk}\dot{q}_k, \tag{13.146}$$

in which

$$c_{ijk} = \frac{1}{2}\left(\frac{\partial m_{ij}}{\partial q_k} + \frac{\partial m_{ik}}{\partial q_j} - \frac{\partial m_{jk}}{\partial q_i} \right). \tag{13.147}$$

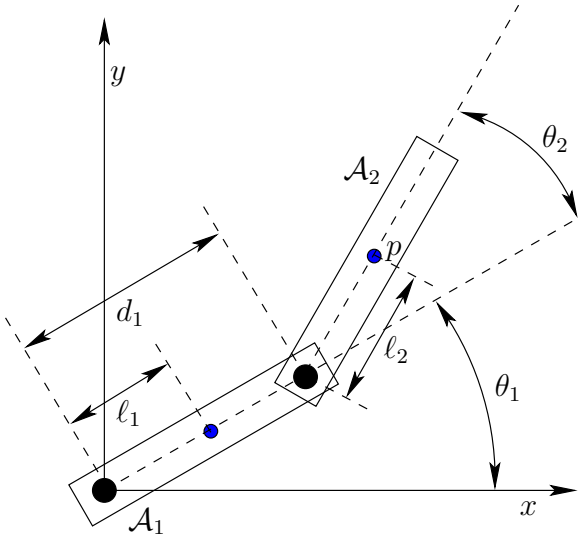


Figure 13.13: Parameter values for a two-link robot with two revolute joints.

This is called a *Christoffel symbol*, and it is obtained from (13.145). Note that $c_{ijk} = c_{ikj}$. Christoffel symbols arise in the study of affine connections in differential geometry and are usually denoted as Γ^i_{jk}. Affine connections provide a way to express acceleration without coordinates, in the same way that the tangent space was expressed without coordinates in Section 8.3.2. For affine connections in differential geometry, see [133]; for their application to mechanics, see [156].

13.4.2.1 Conversion to a phase transition equation

The final step is to convert the equations into phase space form. A $2n$-dimensional phase vector is introduced as $x = (q, \dot{q})$. The task is to obtain $\dot{x} = f(x, u)$, which represents $2n$ scalar equations. The first n equations are $\dot{x}_i = x_{n+i}$ for i from 1 to n. The final n equations are obtained by solving for $\ddot{q}$.

Suppose that the general form in (13.142) is used. Solving for $\ddot{q}$ yields

$$\ddot{q} = M(q)^{-1}(u - C(q, \dot{q})\dot{q} - g(q)). \tag{13.148}$$

The phase variables are then substituted in a straightforward manner. Each $\ddot{q}_i$ for i from 1 to n becomes $\dot{x}_{n+i}$, and $M(q)$, $C(q, \dot{q})$, and $g(q)$ are expressed in terms of x. This completes the specification of the phase transition equation.

Example 13.13 (Two-Link Manipulator) Figure 13.13 shows a two-link manipulator for which there are two revolute joints and two links, $\mathcal{A}_1$ and $\mathcal{A}_2$. Hence, $\mathcal{C} = \mathbb{S}^1 \times \mathbb{S}^1$. Let $q = (\theta_1, \theta_2)$ denote a configuration. Each of the two joints is controlled by a motor that applies a torque u_i. Let u_1 apply to the base, and let u_2 apply to the joint between $\mathcal{A}_1$ and $\mathcal{A}_2$. Let d_1 be the link length of $\mathcal{A}_1$. Let ℓ_i be the distance from the $\mathcal{A}_i$ origin to its center of mass. For each $\mathcal{A}_i$, let m_i and I_i be its mass and moment of inertia, respectively.

The kinetic energy of $\mathcal{A}_1$ is

$$K_1(\dot{q}) = \tfrac{1}{2} m_1 \ell_1 \dot{\theta}_1^2 + \tfrac{1}{2} I_1 \dot{\theta}_1^2, \tag{13.149}$$

and the potential energy of $\mathcal{A}_1$ is

$$V_1(q) = m_1 g \ell_1 \sin \theta_1. \tag{13.150}$$

The kinetic energy of $\mathcal{A}_2$ is

$$K_2(\dot{q}) = \tfrac{1}{2} p \cdot p + \tfrac{1}{2} I_2 (\dot{\theta}_1 + \dot{\theta}_2)^2, \tag{13.151}$$

in which p denotes the position of the center of mass of $\mathcal{A}_1$ and is given from (3.53) as

$$\begin{aligned} p_1 &= d_1 \cos \theta_1 + \ell_2 \cos \theta_2 \\ p_2 &= d_1 \sin \theta_1 + \ell_2 \sin \theta_2. \end{aligned} \tag{13.152}$$

The potential energy of $\mathcal{A}_2$ is

$$V_2(q) = m_2 g (d_1 \sin \theta_1 + \ell_2 \sin \theta_2). \tag{13.153}$$

At this point, the Lagrangian function can be formed as

$$L(q, \dot{q}) = K_1(\dot{\theta}_1) + K_2(\dot{\theta}_1, \dot{\theta}_2) - V_1(\theta_1) - V_2(\theta_1, \theta_2) \tag{13.154}$$

and inserted into (13.118) to obtain the differential constraints in implicit form, expressed in terms of $\ddot{q}$, $\dot{q}$, and q. Conversion to the phase space is performed by solving the implicit constraints for $\ddot{q}$ and assigning $x = (q, \dot{q})$, in which x is a four-dimensional phase vector.

Rather than performing the computations directly using (13.118), the constraints can be directly determined using (13.140). The terms are

$$M(q) = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \tag{13.155}$$

in which

$$\begin{aligned} m_{11} &= I_1 + m_1 \ell_1^2 + I_2 + m_2 (d_1^2 + \ell_2^2 + 2 d_1 \ell_2 \cos \theta_2) \\ m_{12} &= m_{21} = I_2 + m_2 (\ell_2^2 + d_1 \ell_2 \cos \theta_2) \\ m_{22} &= I_2 + m_2 \ell_2^2, \end{aligned} \tag{13.156}$$

and

$$\begin{aligned}
c_{111} &= \frac{1}{2}\frac{\partial m_{11}}{\partial \theta_1} = 0 \\
c_{112} = c_{121} &= \frac{1}{2}\frac{\partial m_{11}}{\partial \theta_2} = -m_2 \ell_1 \ell_2 p_2 \\
c_{122} &= \frac{\partial m_{12}}{\partial \theta_2} - \frac{1}{2}\frac{\partial m_{22}}{\partial \theta_1} = -m_2 \ell_1 \ell_2 p_2 \\
c_{211} &= \frac{\partial m_{21}}{\partial \theta_1} - \frac{1}{2}\frac{\partial m_{11}}{\partial \theta_2} = m_2 \ell_1 \ell_2 p_2 \\
c_{212} = c_{221} &= \frac{1}{2}\frac{\partial m_{22}}{\partial \theta_1} = 0 \\
c_{222} &= \frac{1}{2}\frac{\partial m_{22}}{\partial \theta_2} = 0.
\end{aligned} \tag{13.157}$$

The final term is defined as

$$\begin{aligned}
g_1 &= (m_1 \ell_1 + m_2 d_1) g p_1 + m_1 \ell_2 p_2 \\
g_2 &= m_2 \ell_2 g p_2.
\end{aligned} \tag{13.158}$$

The dynamics can alternatively be expressed using $M(q)$, $C(q, \dot{q})$, and $g(q)$ in (13.142). The Coriolis matrix is defined using (13.143) to obtain

$$C(q, \dot{q}) = -m_2 \ell_1 \ell_2 p_2 \begin{pmatrix} \dot{\theta}_2 & \dot{\theta}_1 + \dot{\theta}_2 \\ \dot{\theta}_1 & 0 \end{pmatrix}, \tag{13.159}$$

in which p_2 is defined in (13.152) and is a function of q. For convenience, let

$$r = m_2 \ell_1 \ell_2 p_2. \tag{13.160}$$

The resulting expression, which is now a special form of (13.142), is

$$\begin{aligned}
m_{11}\ddot{\theta}_1 + m_{12}\ddot{\theta}_2 - 2r\dot{\theta}_1\dot{\theta}_2 - r\dot{\theta}_2^2 + g_1(q) &= u_1 \\
m_{22}\ddot{\theta}_1 + m_{21}\ddot{\theta}_2 + r\dot{\theta}_1^2 + g_2(q) &= u_2.
\end{aligned} \tag{13.161}$$

The phase transition equation is obtained by letting $x = (\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$ and substituting the state variables into (13.161). The variables $\ddot{\theta}_1$ and $\ddot{\theta}_2$ become $\dot{x}_3$ and $\dot{x}_4$, respectively. The equations must be solved for $\dot{x}_3$ and $\dot{x}_4$. An extension of this model to motors that have gear ratios and nonnegligible mass appears in [856]. ■

The example provided here barely scratches the surface on the possible systems that can be elegantly modeled. Many robotics texts cover cases in which there are more links, different kinds of joints, and frictional forces [366, 725, 856, 907, 994].

The phase transition equation for chains of bodies could alternatively be derived using the Newton-Euler formulation of mechanics. Even though the Lagrangian form is more elegant, the Newton-Euler equations, when expressed recursively, are far more efficient for simulations of multibody dynamical systems [366, 863, 994].

13.4.3 Extensions of the Euler-Lagrange Equations

Several extensions of the Euler-Lagrange equation can be constructed to handle complications that arise in addition to kinetic energy and potential energy in a conservative field. Each extension usually involves adding more terms to (13.129) to account for the new complication. Problems that can be handled in this way are closed kinematic chains, nonholonomic constraints, and nonconservative forces (such as friction).

13.4.3.1 Incorporating velocity constraints

The Lagrangian formulation of Section 13.4.1 can be extended to allow additional constraints placed on q and $\dot{q}$. This is very powerful for developing state transition equations for robots that have closed kinematic chains or wheeled bodies. If there are closed chains, then the configurations may be restricted to lie in a subset of $\mathcal{C}$. If a parameterization of the solution set is possible, then $\mathcal{C}$ can be redefined over the reduced C-space. This is usually not possible, however, because such a parametrization is difficult to obtain, as mentioned in Section 4.4. If there are wheels or other contact-based constraints, such as those in Section 13.1.3, then extra constraints on q and $\dot{q}$ exist. Dynamics can be incorporated into the models of Section 13.1 by extending the Euler-Lagrange equation.

The coming method will be based on Lagrange multipliers. Recall from standard calculus that to optimize a function h defined over $\mathbb{R}^n$, subject to an implicit constraint $g(x) = 0$, it is sufficient to consider only the extrema of

$$h(x) + \lambda g(x), \tag{13.162}$$

in which $\lambda \in \mathbb{R}$ represents a *Lagrange multiplier* [508]. The extrema are found by solving

$$\nabla h(x) + \lambda \nabla g(x) = 0, \tag{13.163}$$

which expresses n equations of the form

$$\frac{\partial h}{\partial x_i} + \lambda \frac{\partial g}{\partial x_i} = 0. \tag{13.164}$$

The same principle applies for handling velocity constraints on $\mathcal{C}$.

Suppose that there are velocity constraints on $\mathcal{C}$ as considered in Section 13.1. Consider implicit constraints, in which there are k equations of the form $g_i(q, \dot{q}) = 0$ for i from 1 to k. Parametric constraints can be handled as a special case of implicit constraints by writing

$$g_i(q, \dot{q}) = \dot{q}_i - f_i(q, u) = 0. \tag{13.165}$$

For any constraints that contain actions u, no extra difficulties arise. Each u_i is treated as a constant in the following analysis. Therefore, action variables will not be explicitly named in the expressions.

As before, assume time-invariant dynamics (see [789] for the time-varying case). Starting with $L(q, \dot{q})$ defined using (13.130), let the new criterion be

$$L_c(q, \dot{q}, \lambda) = L(q, \dot{q}) + \sum_{i=1}^{k} \lambda_i g_i(q, \dot{q}). \tag{13.166}$$

A functional Φ_c is defined by substituting L_c for L in (13.114).

The extremals of Φ_c are given by n equations,

$$\frac{d}{dt}\frac{\partial L_c}{\partial \dot{q}_i} - \frac{\partial L_c}{\partial q_i} = 0, \tag{13.167}$$

and k equations,

$$\frac{d}{dt}\frac{\partial L_c}{\partial \dot{\lambda}_i} - \frac{\partial L_c}{\partial \lambda_i} = 0. \tag{13.168}$$

The justification for this is the same as for (13.124), except now λ is included. The equations of (13.168) are equivalent to the constraints $g_i(q, \dot{q}) = 0$. The first term of each is zero because $\dot{\lambda}$ does not appear in the constraints, which reduces them to

$$\frac{\partial L_c}{\partial \lambda_i} = 0. \tag{13.169}$$

This already follows from the constraints on extremals of L and the constraints $g_i(q, \dot{q}) = 0$. In (13.167), there are n equations in $n+k$ unknowns. The k Lagrange multipliers can be eliminated by using the k constraints $g_i(q, \dot{q}) = 0$. This corresponds to Lagrange multiplier elimination in standard constrained optimization [508].

The expressions in (13.167) and the constraints $g_i(q, \dot{q})$ may be quite complicated, which makes the determination of a state transition equation challenging. General forms are given in Section 3.8 of [789]. An important special case will be considered here. Suppose that the constraints are Pfaffian,

$$g_i(q, \dot{q}) = \sum_{j=1}^{n} g_{ij}(q)\dot{q}_j = 0, \tag{13.170}$$

as introduced in Section 13.1. This includes the nonholonomic velocity constraints due to wheeled vehicles, which were presented in Section 13.1.2. Furthermore, this includes the special case of constraints of the form $g_i(q) = 0$, which models closed kinematic chains. Such constraints can be differentiated with respect to time to obtain

$$\frac{d}{dt} g_i(q) = \sum_{j=1}^{n} \frac{\partial g_i}{\partial q_j}\dot{q}_j = \sum_{j=1}^{n} g_{ij}(q)\dot{q}_j = 0, \tag{13.171}$$

which is in the Pfaffian form. This enables the dynamics of closed chains, considered in Section 4.4, to be expressed without even having a parametrization of

the subset of $\mathcal{C}$ that satisfies the closure constraints. Starting in implicit form, differentiation is required to convert them into the Pfaffian form.

For the important case of Pfaffian constraints, (13.167) simplifies to

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} + \sum_{j=1}^{k} \lambda_j g_{ji}(q) = 0, \tag{13.172}$$

The Pfaffian constraints can be used to eliminate the Lagrange multipliers, if desired. Note that g_{ji} represents the ith term of the jth Pfaffian constraint. An action variable u_i can be placed on the right side of each constraint, if desired.

Equation (13.172) often appears instead as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \sum_{l=1}^{k} \lambda_j g_{ji}(q, \dot{q}), \tag{13.173}$$

which is an alternative but equivalent expression of constraints because the Lagrange multipliers can be negated without affecting the existence of extremals. In this case, a nice interpretation due to D'Alembert can be given. Expressions that appear on the right of (13.173) can be considered as actions, as mentioned in Section 13.4.1. As stated previously, such actions are called generalized forces in mechanics. The *principle of virtual work* is obtained by integrating the reaction forces needed to maintain the constraints. These reaction forces are precisely given on the right side of (13.173). Due to the cancellation of forces, no true work is done by the constraints (if there is no friction).

Example 13.14 (A Particle on a Sphere) Suppose that a particle travels on a unit sphere without friction or gravity. Let $(q_1, q_2, q_3) \in \mathbb{R}^3$ denote the position of the point. The Lagrangian function is the kinetic energy,

$$L(q, \dot{q}) = \tfrac{1}{2} m(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2), \tag{13.174}$$

in which m is the particle mass. For simplicity, assume that $m = 2$.

The constraint that the particle must travel on a sphere yields

$$g_1(q) = q_1^2 + q_2^2 + q_3^2 - 1 = 0. \tag{13.175}$$

This can be put into Pfaffian form by time differentiation to obtain

$$2q_1\dot{q}_1 + 2q_2\dot{q}_2 + 2q_3\dot{q}_3 = 0. \tag{13.176}$$

Since $k = 1$, there is a single Lagrange multiplier λ_1. Applying (13.172) yields three equations,

$$\ddot{q}_i - 2q_i\lambda_1 = 0, \tag{13.177}$$

for i from 1 to 3. The generic form of the solution is

$$c_1 q_1 + c_2 q_2 + c_3 q_3 = 0, \tag{13.178}$$

in which the c_i are real-valued constants that can be determined from the initial position of the particle. This represents the equation of a plane through the origin. The intersection of the plane with the sphere is a great circle. This implies that the particle moves between two points by traveling along the great circle. These are the shortest paths (geodesics) on the sphere. ■

The general forms in Section 13.4.2 can be extended to the constrained case. For example, (13.142) generalizes to

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) + G(q)^T\lambda = u, \tag{13.179}$$

in which G is a $n \times k$ matrix that represents all of the g_{ji} Pfaffian coefficients. In this case, the Lagrange multipliers can be computed as [725]

$$\lambda = \big(G(q)M(q)^{-1}G(q)^T\big)^{-1}\, G(q)M(q)^{-1}\big(u - C(q,\dot{q})\dot{q}\big), \tag{13.180}$$

assuming G is time-invariant.

The phase transition equation can be determined in the usual way by performing the required differentiations, defining the $2n$ phase variables, and solving for $\dot{x}$. The result generalizes (13.148).

13.4.3.2 Nonconservative forces

The Lagrangian formulation has been extended so far to handle constraints on $\mathcal{C}$ that lower the dimension of the tangent space. The formulation can also be extended to allow nonconservative forces. The most common and important example in mechanical systems is friction. The details of friction models will not be covered here; see [681]. As examples, friction can arise when bodies come into contact, as in the joints of a robot manipulator, and as bodies move through a fluid, such as air or water. The nonconservative forces can be expressed as additional generalized forces, expressed in an $n \times 1$ vector of the form $B(q,\dot{q})$. Suppose that an action vector is also permitted. The modified Euler-Lagrange equation then becomes

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = u - B(\dot{q}, q). \tag{13.181}$$

A common extension to (13.142) is

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + N(q,\dot{q}) = u, \tag{13.182}$$

in which $N(q,\dot{q})$ generalizes $g(q)$ to include nonconservative forces. This can be generalized even further to include Pfaffian constraints and Lagrange multipliers,

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + N(q,\dot{q}) + G(q)^T\lambda = u. \tag{13.183}$$

The Lagrange multipliers become [725]

$$\lambda = \big(G(q)M(q)^{-1}G(q)^T\big)^{-1}\, G(q)M(q)^{-1}\big(u - C(q,\dot{q})\dot{q} - N(q,\dot{q})\big). \tag{13.184}$$

Once again, the phase transition equation can be derived in terms of $2n$ phase variables and generalizes (13.148).

13.4.4 Hamiltonian Mechanics

The Lagrangian formulation of mechanics is the most convenient for determining a state transition equation for a collection of bodies. Once the kinetic and potential energies are determined, the remaining efforts are straightforward computation of derivatives and algebraic manipulation. Hamiltonian mechanics provides an alternative formulation that is closely related to the Lagrangian. Instead of expressing second-order differential constraints on an n-dimensional C-space, it expresses first-order constraints on a $2n$-dimensional phase space. This idea should be familiar from Section 13.2. The new phase space considered here is an example of a *symplectic manifold*, which has many important properties, such as being orientable and having an even number of dimensions [39]. The standard phase vector is defined as $x = (q, \dot{q})$; however, instead of $\dot{q}$, n variables will be introduced and denoted as p. Thus, a transformation exists between $(q, \dot{q})$ and (p, q). The p variables are related to the configuration variables through a special function over the phase space called the *Hamiltonian*. Although the Hamiltonian formulation usually does not help in the determination of $\dot{x} = f(x, u)$, it is covered here because its generalization to optimal control problems is quite powerful. This generalization is called Pontryagin's minimum principle and is covered in Section 15.2.3. In the context of mechanics, it provides a general expression of energy conservation laws, which aids in proving many theoretical results [39, 397].

The relationship between $(q, \dot{q})$ and (p, q) can be obtained by using the *Legendre transformation* [39, 397]. Consider a real-valued function f of two variables, $x, y \in \mathbb{R}$. Its *total differential* [508] is

$$df = u\, dx + v\, dy, \tag{13.185}$$

in which

$$u = \frac{\partial f}{\partial x} \qquad \text{and} \qquad v = \frac{\partial f}{\partial y}. \tag{13.186}$$

Consider constructing a total differential that depends on du and dy, instead of dx and dy. Let g be a function of u and y defined as

$$g(u, y) = ux - f. \tag{13.187}$$

The total differential of g is

$$dg = x\, du + u\, dx - df. \tag{13.188}$$

Using (13.185) to express df, this simplifies to

$$dg = x\, du - v\, dy. \tag{13.189}$$

The x and v variables are now interpreted as

$$x = \frac{\partial g}{\partial u} \qquad\qquad v = -\frac{\partial g}{\partial y}, \tag{13.190}$$

which appear to be a kind of inversion of (13.186). This idea will be extended to vector form to arrive the Hamiltonian formulation.

Assume that the dynamics do not depend on the particular time (the extension to time-varying dynamics is not difficult; see [39, 397]). Let $L(q, \dot{q})$ be the Lagrangian function defined (13.129). Let $p \in \mathbb{R}^n$ represent a *generalized momentum* vector (or *adjoint variables*), which serves the same purpose as u in (13.185). Each p_i is defined as

$$p_i = \frac{\partial L}{\partial \dot{q}_i}. \tag{13.191}$$

In some literature, p is instead denoted as λ because it can also be interpreted as a vector of Lagrange multipliers. The *Hamiltonian function* is defined as

$$H(p, q) = p \cdot \dot{q} - L(q, \dot{q}) = \sum_{i=1}^{n} p_i \dot{q}_i - L(q, \dot{q}) \tag{13.192}$$

and can be interpreted as the total energy of a conservative system [397]. This is a vector-based extension of (13.187) in which L and H replace f and g, respectively. Also, p and q are the vector versions of u and x, respectively.

Considered as a function of p and q only, the total differential of H is

$$dH = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} dp_i + \sum_{i=1}^{n} \frac{\partial H}{\partial q_i} dq_i. \tag{13.193}$$

Using (13.192), dH can be expressed as

$$dH = \sum_{i=1}^{n} \dot{q}_i \, dp_i + \sum_{i=1}^{n} p_i \, d\dot{q}_i - \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i - \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} dq_i. \tag{13.194}$$

The $d\dot{q}_i$ terms all cancel by using (13.191), to obtain

$$dH = \sum_{i=1}^{n} \dot{q}_i \, dp_i - \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} dq_i. \tag{13.195}$$

Using (13.118),

$$\dot{p} = \frac{\partial L}{\partial q_i}. \tag{13.196}$$

This implies that

$$dH = \sum_{i=1}^{n} \dot{q}_i \, dp_i - \sum_{i=1}^{n} \dot{p}_i \, dq_i. \tag{13.197}$$

Equating (13.197) and (13.193) yields $2n$ equations called *Hamilton's equations*:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad\qquad \dot{p}_i = \frac{\partial H}{\partial q_i}, \tag{13.198}$$

for each i from 1 to n. These equations are analogous to (13.190).

Hamilton's equations are equivalent to the Euler-Lagrange equation. Extremals in both cases yield equivalent differential constraints. The difference is that the Lagrangian formulation uses $(q, \dot{q})$ and the Hamiltonian uses (p, q). The Hamiltonian results in first-order partial differential equations. It was assumed here that the dynamics are time-invariant and the motions occur in a conservative field. In this case, $dH = 0$, which corresponds to conservation of total energy. In the time-varying case, the additional equation $\partial H/\partial t = -\partial L/\partial t$ appears along with Hamilton's equations. As stated previously, Hamilton's equations are primarily of interest in establishing basic results in theoretical mechanics, as opposed to determining the motions of particular systems. For example, the Hamiltonian is used to establish Louisville's theorem, which states that phase flows preserve volume, implying that a Hamiltonian system cannot be asymptotically stable [39]. Asymptotic stability is covered in Section 15.1.1. Pontryagin's minimum principle, an extension of Hamilton's equations to optimal control theory, is covered in 15.2.3.

13.5 Multiple Decision Makers

Differential models can be extended to model the interaction of multiple decision makers. This leads to continuous-time extensions of sequential decision making, from Formulation 10.1, and sequential games, from Formulation 10.4. A differential version of the state transition equation can be made for these extensions.

13.5.1 Differential Decision Making

To make a *differential game against nature* that extends Formulation 10.1 to continuous time, suppose that nature actions $\theta(t)$ are chosen from Θ. A differential model can be defined as

$$\dot{x} = f(x, u, \theta). \tag{13.199}$$

The state space X and action space U are used in the same way as throughout this chapter. The difference only comes in the state transition equation. State-dependent nature action spaces may also be used.

As observed repeatedly throughout Part III, nature can be modeled nondeterministically or probabilistically. In the nondeterministic case, (13.199) is equivalent to a *differential inclusion* [53]:

$$\dot{x} \in \{\dot{x}' \mid \exists \theta \in \Theta \text{ such that } \dot{x}' = f(x, u, \theta)\}. \tag{13.200}$$

Possible future values for $\dot{x}$ can be computed using forward projections. Reachable sets, which will be introduced in Section 14.2.1, can be defined that characterize the evolution of future possible states over time. Plans constructed under this model usually use worst-case analysis.

Example 13.15 (Nondeterministic Forward Projection) As a simple example of using (13.199), consider expressing the uncertainty model used in the preimage planning framework of Section 12.5.1.

At each time $t \geq 0$, nature chooses some $\theta \in \Theta(t)$. The state transition equation is

$$\dot{x} = u + \theta. \tag{13.201}$$

The cone shown in Figure 12.45 is just the nondeterministic forward projection under the application of a constant $u \in U$. ■

In the probabilistic case, restrictions must be carefully placed on the nature action trajectory (e.g., a Weiner process [910]). Under such conditions, (13.199) becomes a *stochastic differential equation*. Planning in this case becomes continuous-time stochastic control [567], and the task is to optimize the expected cost.

Example 13.16 (A Simple Car and Nature) Uncertainty can be introduced into any of the models of this chapter. For example, recall the simple car, (13.15). Suppose that nature interferes with the steering action so that it is not precisely known in which direction the car will drive. Let $\Theta = [-\theta_{max}, \theta_{max}]$, in which $\theta_{max} \in (0, \pi/2)$ represents the maximum amount of steering angle error that can be caused by nature. The simple-car model can be modified to account for this error as

$$\begin{aligned} \dot{x} &= u_s \cos\theta \\ \dot{y} &= u_s \sin\theta \\ \dot{\theta} &= \frac{u_s}{L} \tan(u_\phi + \gamma), \end{aligned} \tag{13.202}$$

in which the domain of tan must be extended to $\mathbb{R}$ or other suitable restrictions must be imposed. At each time t, a nature action[12] $\gamma \in \Theta$ causes the true heading of the car to be perturbed from the commanded direction u_ϕ. Under nondeterministic uncertainty, the maximum amount that the car deviates from the commanded direction must be determined by the planning algorithm. A probability density function $p(\gamma)$ can be assigned to obtain a probabilistic model. When integrated over time, (13.202) yields probability density functions over future car configurations [1004]. ■

In a similar way, parameters that account for nature can be introduced virtually anywhere in the models of this chapter. Some errors may be systematic, which reflect mistakes or simplifications made in the modeling process. These correspond to a constant nature action applied at the outset. In this case, nature is not allowed to vary its action over time. Other errors could correspond to noise, which is expected to yield different nature actions over time.

[12]The notation γ is used instead of θ to avoid conflicting with the car orientation variable θ in this particular example.

13.5.2 Differential Game Theory

The extension of sequential game theory to the continuous-time case is called *differential game theory* (or *dynamic game theory* [59]), a subject introduced by Isaacs [477]. All of the variants considered in Sections 9.3, 9.4, 10.5 are possible:

1. There may be any number of players.
2. The game may be zero-sum or nonzero-sum.
3. The state may or may not be known. If the state is unknown, then interesting I-spaces arise, similar to those of Section 11.7.
4. Nature can interfere with the game.
5. Different equilibrium concepts, such as saddle points and Nash equilibria, can be defined.

See [59] for a thorough overview of differential games. Two players, P_1 and P_2, can be engaged in a *differential game* in which each has a continuous set of actions. Let U and V denote the action spaces of P_1 and P_2, respectively. A state transition equation can be defined as

$$\dot{x} = f(x, u, v), \tag{13.203}$$

in which x is the state, $u \in U$, and $v \in V$.

Linear differential games are an important family of games because many techniques from optimal control theory can be extended to solve them [59].

Example 13.17 (Linear Differential Games) The linear system model (13.37) can be extended to incorporate two players. Let $X = \mathbb{R}^n$ be a phase space. Let $U = \mathbb{R}^{m_1}$ and $V = \mathbb{R}^{m_2}$ be an action spaces for $m_1, m_2 \leq n$. A *linear differential game* is expressed as

$$\dot{x} = Ax + Bu + Cv, \tag{13.204}$$

in which A, B, and C are constant, real-valued matrices of dimensions $n \times n$, $n \times m_1$, and $n \times m_2$, respectively. The particular solution to such games depends on the cost functional and desired equilibrium concept. For the case of a quadratic cost, closed-form solutions exist. These extend techniques that are developed for linear systems with one decision maker; see Section 15.2.2 and [59].

The original work of Isaacs [477] contains many interesting examples of *pursuit-evasion differential games*. One of the most famous is described next.

Example 13.18 (Homicidal Chauffeur) In the *homicidal chauffeur* game, the pursuer is a Dubins car and the evader is a point robot that can translate in any direction. Both exist in the same world, $\mathcal{W} = \mathbb{R}^2$. The speeds of the car and robot are s_1 and s_2, respectively. It is assumed that $|s_1| > |s_2|$, which means that the pursuer moves faster than the evader. The transition equation is given

by extending (13.15) to include two state variables that account for the robot position:

$$\begin{aligned} \dot{x}_1 &= s_1 \cos\theta_1 & \qquad \dot{x}_2 &= s_2 \cos v \\ \dot{y}_1 &= s_1 \sin\theta_1 & \qquad \dot{y}_2 &= s_2 \sin v \\ \dot{\theta}_1 &= \frac{s_1}{L} \tan u_\phi. \end{aligned} \tag{13.205}$$

The state space is X is $\mathbb{R}^4 \times \mathbb{S}^1$, and the action spaces are $U = [-\phi_{max}, \phi_{max}]$ and $V = [0, 2\pi)$.

The task is to determine whether the pursuer can come within some prescribed distance ϵ of the evader:

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 < \epsilon^2. \tag{13.206}$$

If this occurs, then the pursuer wins; otherwise, the evader wins. The solution depends on the L, s_1, s_2, ϵ, and the initial state. Even though the pursuer moves faster, the evader may escape because it does not have a limited turning radius. For given values of L, s_1, s_2, and ϵ, the state space X can be partitioned into two regions that correspond to whether the pursuer or evader wins [59, 477]. To gain some intuition about how this partition may appear, imagine the motions that a bullfighter must make to avoid a fast, charging bull (yes, bulls behave very much like a fast Dubins car when provoked). ■

Another interesting pursuit-evasion game arises in the case of one car attempting to intercept another [694].

Example 13.19 (A Game of Two Cars) Imagine that there are two simple cars that move in the same world, $\mathcal{W} = \mathbb{R}^2$. Each has a transition equation given by (13.15). The state transition equation for the game is

$$\begin{aligned} \dot{x}_1 &= u_s \cos\theta_1 & \qquad \dot{x}_2 &= v_s \cos\theta_2 \\ \dot{y}_1 &= u_s \sin\theta_1 & \qquad \dot{y}_2 &= v_s \sin\theta_2 \\ \dot{\theta}_1 &= \frac{u_s}{L_1} \tan u_\phi & \qquad \dot{\theta}_2 &= \frac{v_s}{L_2} \tan v_\phi. \end{aligned} \tag{13.207}$$

The pursuit-evasion game becomes very interesting if both players are restricted to be Dubins cars. ■

Further Reading

This chapter was synthesized from numerous sources. Many important, related subjects were omitted. For some mechanics of bodies in contact and manipulation in general, see [681]. Three-dimensional vehicle models were avoided because they are complicated

by $SO(3)$; see [433]. For computational issues associated with simulating dynamical systems, see [247, 863].

For further reading on velocity constraints on the C-space, see [596, 725] and Sections 15.3 to 15.5. For more problems involving rolling spheres, see [527] and references therein. The rolling-ball problem is sometimes referred to as the Chaplygin ball. A nonholonomic manipulator constructed from rolling-ball joints was developed and analyzed in [729]. The kinematics of curved bodies in contact was studied in [632, 716]. For motion planning in this context, see [101, 103, 223, 676]. Other interesting nonholonomic systems include the snakeboard [473, 629], roller racer [556], rollerblader [214], Trikke [213], and examples in [112] (e.g., the Chaplygin sled).

Phase space representations are a basic part of differential equations, physics, and control theory; see [44, 192].

Further reading in mechanics is somewhat complicated by two different levels of treatment. Classical mechanics texts do not base the subject on differential geometry, which results in cumbersome formulations and unusual terminology (e.g., generalized coordinates). Modern mechanics texts overcome this problem by cleanly formulating everything in terms of geodesics on Riemannian manifolds; however, this may be more difficult to absorb for readers without background in differential geometry. An excellent source for modern mechanics is [39]. One of the most famous texts for classical mechanics is [397]. For an on-line book that covers the calculus of variations, including constrained Lagrangians, see [790]. The constrained Lagrangian presentation is based on Chapter 3 of [789], Section 2.4 of [397], and parts of [405]. Integral constraints on the Lagrangian are covered in [790], in addition to algebraic and differential constraints. Lagrangian mechanics under inequality constraints is considered in [789]. The presentation of the Hamiltonian in Section 13.4.4 is based on Chapter 7 of [397] and Section 15 of [39]. For advanced, modern treatments of mechanics in the language of affine connections and Christoffel symbols, see [3, 156, 677]. Another source, which is also heavily illustrated, is [359]. For further reading on robot dynamics, see [30, 204, 725, 856, 907, 994]. For dynamics of automobiles, see [389].

For further reading on differential game theory, primary sources are [59, 423, 477]; see also [34, 57, 783, 985, 991, 992, 993, 997]. Lower bounds for the algorithmic complexity of pursuit-evasion differential games are presented in [821].

Exercises

1. Let $\mathcal{C} = \mathbb{R}^4$. There are two Pfaffian constraints, $\dot{q}_1 + \dot{q}_2 + \dot{q}_3 + \dot{q}_4 = 0$ and $\dot{q}_2 - \dot{q}_4 = 0$. Determine the appropriate number of action variables and express the differential constraints in the form $\dot{q} = f(q, u)$.

2. Introduce a phase space and convert $2\ddot{y} - 10\dot{y}^2 + 5y = 0$ into the form $\dot{x} = f(x)$.

3. Introduce a phase space and convert $y^{(4)} + y = 0$ into the form $\dot{x} = f(x)$.

4. Derive the configuration transition equation (13.19) for a car pulling trailers.

5. Use the main idea of Section 13.2.4 to develop a smooth-steering extension of the car pulling trailers, (13.19).

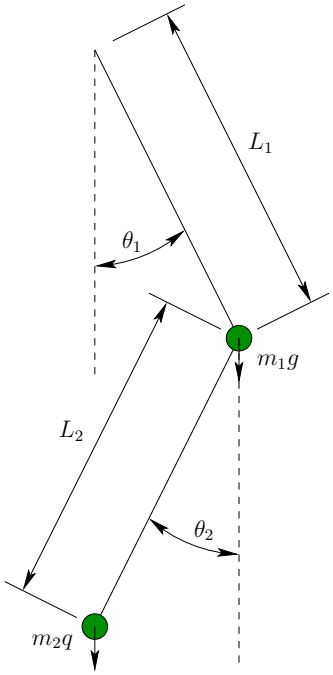


Figure 13.14: A double pendulum.

6. Suppose that two identical differential-drive robots are connected together at their centers with a rigid bar of length d. The robots are attached at each end of the rod, and each attachment forms a revolute joint. There are four wheels to control; however, some combinations of wheel rotations cause skidding. Assuming that skidding is not allowed, develop a motion model of the form $\dot{q} = f(q, u)$, in which $\mathcal{C}$ and U are chosen to reflect the true degrees of freedom.

7. Extend the lunar lander model to a general rigid body with a thruster that does not apply forces through the center of mass.

8. Develop a model for a 3D rotating rigid body fired out of a canon at a specified angle above level ground under gravity. Suppose that thrusters are placed on the body, enabling it to be controlled before it impacts the ground. Develop general phase transition equations.

9. Add gravity with respect to q_2 in Example 13.12 and derive the new state transition equation using the Euler-Lagrange equation.

10. Use the constrained Lagrangian to derive the equations of motion of the pendulum in Example 13.8.

11. Define a phase space, and determine an equation of the form $\dot{x} = f(x)$ for the double pendulum shown in Figure 13.14.

12. Extend Example 13.13 to obtain the dynamics of a three-link manipulator. The third link, $\mathcal{A}_3$, is attached to the other two by a revolute joint. The new parameters are θ_3, d_2, ℓ_3, m_3, and I_3.

13. Solve Example 13.14 by parameterizing the sphere with standard spherical coordinates and using the unconstrained Lagrangian. Verify that the same answer is obtained.

14. Convert the equations in (13.161) into phase space form, to obtain the phase transition equation in the form $\dot{x} = f(x, u)$. Express the right side of the equation in terms of the basic parameters, such as mass, moment of inertia, and lengths.

15. Define the Hamiltonian for a free-floating 2D rigid body under gravity and develop Hamilton's equations.

Implementations

16. Make a 3D spacecraft (rigid-body) simulator that allows any number of binary thrusters to be placed in any position and orientation.

17. Make a simulator for the two-link manipulator in Example 13.13.